



저작자표시-비영리-변경금지 2.0 대한민국

이용자는 아래의 조건을 따르는 경우에 한하여 자유롭게

- 이 저작물을 복제, 배포, 전송, 전시, 공연 및 방송할 수 있습니다.

다음과 같은 조건을 따라야 합니다:



저작자표시. 귀하는 원저작자를 표시하여야 합니다.



비영리. 귀하는 이 저작물을 영리 목적으로 이용할 수 없습니다.



변경금지. 귀하는 이 저작물을 개작, 변형 또는 가공할 수 없습니다.

- 귀하는, 이 저작물의 재이용이나 배포의 경우, 이 저작물에 적용된 이용허락조건을 명확하게 나타내어야 합니다.
- 저작권자로부터 별도의 허가를 받으면 이러한 조건들은 적용되지 않습니다.

저작권법에 따른 이용자의 권리는 위의 내용에 의하여 영향을 받지 않습니다.

이것은 [이용허락규약\(Legal Code\)](#)을 이해하기 쉽게 요약한 것입니다.

[Disclaimer](#)

Master's Thesis

An Efficient Numerical Approach for Solving Caputo
Fractional Nonlinear Two-Point Boundary Value
Problems

Junseo Lee

Department of Mathematical Sciences

Graduate School of UNIST

2019

An efficient Numerical Approach for Solving
Nonlinear Two-Point Boundary Value Problems with
a Caputo Fractional Derivatives

Junseo Lee

Department of Mathematical Sciences

Graduate School of UNIST

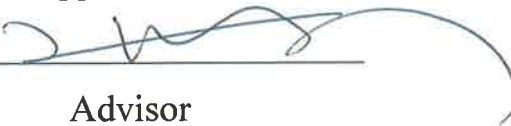
An Efficient Numerical Approach for Solving Caputo Fractional Nonlinear Two-Point Boundary Value Problems

A thesis/dissertation
submitted to the Graduate School of UNIST
in partial fulfillment of the
requirements for the degree of
Master of Science

Junseo Lee

06/24/2020

Approved by



Advisor

Bongsoo Jang

An Efficient Numerical Approach for Solving Caputo Fractional Nonlinear Two-Point Boundary Value Problems

Junseo Lee

This certifies that the thesis/dissertation of Junseo Lee is approved.

06/24/2020

signature



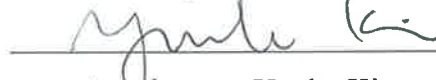
Advisor: Bongsoo Jang

signature



typed name: Changyeol Jung

signature



typed name: Yunho Kim

signature

Abstract

This paper proposed a new type of numerical scheme for solving Multi-term Fractional Nonlinear Differential Equation with Two Point Boundary Value Problems(FBVP).

The proposed methods take lower computational cost than conventional methods. In new techniques, the FBVP transposes to the System of Fractional Nonlinear Initial Value Problems(FIVP). In order to solve the system of FIVP, the Higher-Order Predictor-Corrector Method [1](HOM) is applied. Moreover, we employ shooting method based on Newton's and Halley's methods to approximate the unknown initial values of the system. Several numerical experiments show that the proposed methods give the same rate of convergence in the HOM.

Keywords : Caputo fractional derivative, Fractional differential equation, Fractional order boundary value problem, predictor-corrector methods, Nonlinear shooting methods.

Contents

I	Introduction	1
1.1	Model Problem	1
1.2	Preliminaries	2
II	Fractional Two-Point Boundary Value Problem	6
2.1	Description about Fractional Boundary Value Problem	6
2.2	Numerical Methods	8
2.3	Numerical Results	14
III	Fractional Two-Point Boundary Value Problem with 1st order Derivative Term .	36
3.1	Description of Fractional Boundary Value Problem containing 1st order Derivative	37
3.2	Numerical Results	41
IV	Application	47
4.1	Variable order Fractional Two-Point Boundary Value Problem	47
V	Conclusion	51
	References	53
	Acknowledgements	55

I Introduction

1.1 Model Problem

Fractional differential equations recently receive attention. Because, it can more flexibly express many mathematical models than ordinary differential equation. Thus, many physics and engineering parts use that concepts. For problems like biology [2] [3] [4], earthquake modeling [5], Heat flow and diffusion [6], Thermal wave [7] and Neural Network [8] [9] to name but a few. Thus, various fields need solvers for diverse fractional type of differential equations. This research introduce how to solve FBVP with Caputo operator.

$$\begin{cases} {}^C D_{t_0,t}^{\alpha_2} y(t) = f(t, y, {}^C D_{t_0,t}^{\alpha_1}) \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b} \end{cases} \quad (I.1)$$

where $0 < \alpha_1 < 1$, $1 < \alpha_2 < 2$, $\alpha_1, \alpha_2 \in \mathbb{R}$, g be a linear function and γ_t is for some real number.

In general method construct a matrix for imposing FBVP. But, due to the non-local property, it takes too many computational costs to solve a matrix equation at each time. Thus, this paper suggest new numerical scheme that transform main problem to a system of FIVP. However, that initial value problems doesn't have a whole exact initial value, so if you want to solve that problem, then you must approximate that value which most well solve the system. Thus, this research will use shooting method for renewing an approximation of a initial value problem. That Shooting Methods are Newton's and Halley's Method which have second and third order. The conventional methods basically use predictor-corrector method. So, each system need to find predictor and corrector. But the nonlinear system which comes from original boundary value problem has a special structure that is each equation of transformed system influence other. Therefore, the paper try to use that structure.

1.2 Preliminaries

This section briefly present some definitions and theorems of the fractional calculus which is enough to understand contents of this paper. And it will introduce How to solve Fractional differential equation with initial value problem.

1.2.1 Fractional Calculus

Some basics of the fractional calculus for the later parts of this paper will be introduced. The origins of fractional calculus begin at 17th century. Leibniz and L'Hospital discussed about “what is $\frac{d^{1/2}y}{dx^{1/2}}$?”. After that discussion, various engineers and mathematicians suggest diverse types of fractional differential operator, such as “Riemann-Liouville operator” [10], “Grünwald-Letnikov operator” [10], “Caputo operator” [10], “Caputo-Fabrizio operator” [11] and “Atangana-Baleanu operator” [12]. In this paper we only consider the Caputo operator. Below content will begin with special functions and a brief motivation for the Caputo fractional derivative.

For fractional calculus start from classical result of relation between differential and integration which is called ‘Fundamental Theorem of Calculus’.

Now let's define some notation for research's convenient.

Definition 1.2.1. Let's denote D as the operator that maps a differentiable function onto its derivative,

$$Dy(x) := y'(x) \quad (\text{I.2})$$

and notation J is the operator that maps a function f , assumed to be Riemann integrable on the compact interval $[a, b]$, onto its primitive centered at a

$$J_a y(t) := \int_a^t y(\tau) d\tau \quad (\text{I.3})$$

for $n \in \mathbb{N}$, D^n and J_a^n notation will be used. That means n -fold iterates of D and J_a respectively.

$$D^1 := D, J_a^1 := J_a, D^n := DD^{n-1}, \text{ and } J_a^n = J_a J_a^{n-1} \quad (\text{I.4})$$

for $n \geq 2$.

Following the above definition, let's begin with the integral operator J_a^n . In the case $n \in \mathbb{N}$, it can be replace by the following explicite formula.

Lemma 1.2.1. Let f be Riemann integrable on $[a, b]$. Then, for $a \leq t \leq b$ and $n \in \mathbb{N}$, we have

$$J_a^n y(t) = \frac{1}{(n-1)!} \int_a^t (t-\tau)^{n-1} y(\tau) d\tau. \quad (\text{I.5})$$

Theorem 1.2.1. Fundamental Theorem of Calculus Let $f \in C[a, b]$ is real valued function. and let $F : [a, b] \rightarrow \mathbb{R}$ is defined by

$$F(x) := \int_a^x f(t) dt. \quad (\text{I.6})$$

Then, F is differentiable and

$$F' = f \quad (\text{I.7})$$

In this theorem shows that differential and integration operator have really close relation. When fractional derivative is induced, that relation be a very important point. Moreover, it is an immediate consequence of the fundamental theorem that the following relation holds for the operators D and J_a .

Lemma 1.2.2. *Let, $m, n \in \mathbb{N}$ such that $m > n$, and let f be a function having a continuous n th derivative on the interval $[a, b]$. Then,*

$$D^n y = D^m J_a^{m-n} y. \quad (\text{I.8})$$

Now, let's define a special function. "Gamma function" is really many used in fractional calculus and also in this paper.

Definition 1.2.2. *Gamma function $\Gamma : (0, \infty) \rightarrow \mathbb{R}$, defined by*

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt, \quad (\text{I.9})$$

is called Euler's Gamma function

Now, the following definition seems rather natural.

Definition 1.2.3. *Riemann-Liouville Integrals Let $\alpha \in \mathbb{R}_+$. The operator J_a^α , defined on $L_1[a, b]$ by*

$$J_a^\alpha y(t) := \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} y(\tau) d\tau \quad (\text{I.10})$$

for $a \leq t \leq b$, is called the Riemann-Liouville fractional integral operator of order α .

For $\alpha = 0$, $J_a^0 = I$, the identity operator. And Riemann-Liouville integral operator has following property.

Theorem 1.2.2. *Let $\alpha, \beta \in \mathbb{R}_+$, $\alpha, \beta \geq 0$ and $y \in L_1[a, b]$. Then,*

$$J_a^\alpha J_a^\beta y(t) = J_a^{\alpha+\beta} y(t) \quad (\text{I.11})$$

holds almost everywhere on $[a, b]$. If additionally $y \in C[a, b]$ or $\alpha + \beta \geq 1$, then the identity holds everywhere on $[a, b]$.

To motive the definition coming up, we recall **Lemma 1.2.2** that states the identity.

$$D^n y = D^m J_a^{m-n} y.$$

where m and n were integers such that $m > n$. Now assume that n is not an integer. Then m still can be chosen an integer m such that $m > n$. That induced new definition.

Definition 1.2.4. *Riemann-Liouville operator let $\alpha \in \mathbb{R}_+$. The operator ${}^{RL}D_a^\alpha$ is defined by*

$${}^{RL}D_a^\alpha y(t) := D^{\lceil \alpha \rceil} J_a^{\lceil \alpha \rceil - \alpha} y(t) = \frac{1}{\Gamma(\lceil \alpha \rceil - \alpha)} \frac{d^{\lceil \alpha \rceil}}{dt^{\lceil \alpha \rceil}} \int_a^t (t - \tau)^{\lceil \alpha \rceil - \alpha} y(\tau) d\tau. \quad (\text{I.12})$$

where $\lceil \cdot \rceil$ is ceiling function, and $\lfloor \cdot \rfloor$ is a floor function. if $\alpha \in \mathbb{N}$ then, ${}^{RL}D_a^\alpha := D^\alpha$.

But, it turns out that the Riemann-Liouville derivative have certain disadvantages when trying to model real-world phenomena with fractional differential equations. So there are many alternative operator of the Riemann-Liouville operator. For instance “Caputo operator”, “Caputo-Fabrizio operator” and “Atangana-Baleanu operator”. And this paper focus on “Caputo operator”.

Definition 1.2.5. *Caputo operator let $\alpha \in \mathbb{R}_+$. The operator ${}^C D_a^\alpha$ is defined by*

$${}^C D_a^\alpha y(t) := J_a^{[\alpha]-\alpha} D^{[\alpha]} y(t) = \frac{1}{\Gamma([\alpha] - \alpha)} \int_a^t (t - \tau)^{[\alpha]-\alpha} y^{([\alpha])}(\tau) d\tau. \quad (\text{I.13})$$

where $\lceil \cdot \rceil$ is a ceiling function, and $\lfloor \cdot \rfloor$ is a floor function. if $\alpha \in \mathbb{N}$ then, ${}^C D_a^\alpha := D^\alpha$.

Then, there is difference between Riemann-Liouville derivative and Caputo derivative.

Theorem 1.2.3. *Let $\alpha \in \mathbb{R}_+$. And assume that $f \in A^{[\alpha]}[a, b]$, ($A^m[a, b]$, $m \in \mathbb{N}$ is the set of functions with an absolutely continuous $(m - 1)$ st derivative.)*

$${}^C D_a^\alpha y(t) = {}^{RL} D_a^\alpha [y - T_{[\alpha]}[y; a]] \quad (\text{I.14})$$

almost everywhere. Here $T_m[f; a]$ denote the Taylor polynomial of degree m for the function y . centered at a ; if m is 0, then $T_0[y, a] = 0$.

And Caputo operator satisfies additive property.

Theorem 1.2.4. *Let $f \in C^k[a, b]$ for some $a < b$ and some $k \in \mathbb{N}$. Moreover let $\alpha, \beta > 0$ be such that there exists some $l \in \mathbb{N}$ with $l \leq k$ and $\alpha, \alpha + \beta \in [l - 1, l]$. Then,*

$$D_a^\beta D_a^\alpha y(t) = D_a^{\alpha+\beta} y(t), \quad (\text{I.15})$$

This paper will denote $D_a^\alpha y(t) \equiv {}^C D_a^\alpha y(t)$.

1.2.2 Fractional Initial Value Problem

FIVP is consider many branch. For instance, biological branches like tumor model [3], chicken pox disease model [4]. Second, analysis about heat flow [6]. Third, financial branches like Black-Scholes option pricing equation [13]. Fourth, neural networks which is a part of deep learning also have been researched with a fractional differential equation [9]. And there are more branches that use the initial value problem. Thus, two things will be introduced. First, definition of fractional initial value problem. Second, some theorems for it. They are about existence, equivalent form of solution and uniqueness of that form. Every below theorems or lemmas come from [10].

$$\begin{cases} D_a^\alpha y(t) = f(t, y(t)), & t \in [a, b] \\ y^{(k)}(a) = y_k, & k = 0, 1, \dots, [\alpha] \end{cases} \quad (\text{I.16})$$

Then, the existence of solution for FIVP (I.16) is guaranteed by below theorem.

Theorem 1.2.5. Let $0 < \alpha$ and $y_a^{(0)}, \dots, y^{([\alpha])} \in \mathbb{R}$, $K > 0$ and $h^* > 0$.

Define $G := \left\{ (x, y) \left| x \in [a, b^*], |b^* - a| = h^*, \left| y - \sum_{k=0}^{[\alpha]} (x-a)^k y_a^{(k)} / k! \right| \leq K \right\}$.

and let the function $f : G \rightarrow \mathbb{R}$ be continuous. Beside, define $M := \sup_{(x,z) \in G} |f(x,z)|$ and

$$h := \begin{cases} h^* & \text{if } M = 0 \\ \min \{ h^*, (KT(\alpha+1)/M)^{1/\alpha} \} & \text{else} \end{cases} \quad (\text{I.17})$$

Then, there exists a number b such that $|a - b| = h$ and also exists function $y \in C[a, b]$ solving the initial value problem (I.16).

Existence of solution of (I.16) is guaranteed by Theorem (1.2.5). Then, next Lemma is going to show that what equation is equivalent to solution of (I.16) and that following lemma will be adapted to numerical method for.

Lemma 1.2.3. Assume the hypotheses of Theorem (1.2.5). The function $y \in C[a, b]$ is a solution of the initial value problem (I.16) if and only if it is a solution of the nonlinear Volterra integral equation of the second kind.

$$y(t) = \sum_{k=0}^{[\alpha]} \frac{(x-a)^k}{k!} y_k + \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \quad (\text{I.18})$$

Next, following theorem guarantee uniqueness of solution.

Theorem 1.2.6. Let $0 < \alpha$ and $y_a^{(0)}, \dots, y^{([\alpha])} \in \mathbb{R}$, $K > 0$ and $h^* > 0$. Define the set G as in Theorem (1.2.5) and let the function $f : G \rightarrow \mathbb{R}$ continuous and fulfill a Lipschitz condition with respect to the second variable, i.e

$$|f(x, y_1) - f(x, y_2)| \leq L|y_1 - y_2| \quad (\text{I.19})$$

with some $L > 0$ independent of x, y_1 and y_2 . Then, denoting h as in Theorem (1.2.5). there exists a uniquely defined function $y \in C[a, b]$ solving the initial value problem (I.16).

II Fractional Two-Point Boundary Value Problem

The main part of this research suggest a new numerical scheme for solving the following FBVP:

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y, D_a^{\alpha_1} y(t)) \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b}, \end{cases} \quad (\text{II.1})$$

where $1 < \alpha_2 < 2$, $0 < \alpha_1 < 1$, $\alpha_2, \alpha_1 \in \mathbb{R}$ and g be linear function.

If conventional method is used for solving above problem then it need many computational cost to solve a dense matrix and multi-dimensional solver. By the way, there are some theorems for FBVP to transpose system of FIVP. Then, the numerical method for it need less computational cost than conventional one for fractional boundary value problem. However, original FBVP doesn't provide a initial value $y(a)$ or $y'(a)$ data directly. Thus, to approximate that data, shooting method will be used. For examples are Newton's or Halley's methods. And that shooting methods are iterated when the error between result given data is reduced. "Figure 1" will be show outline of above explanation.

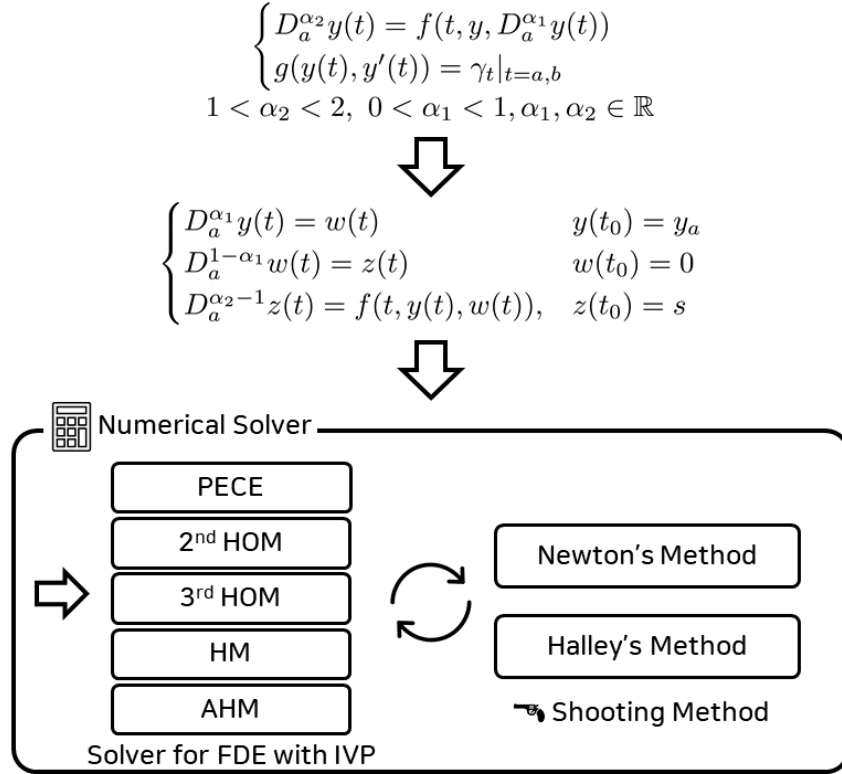


Figure 1: Diagram about Solving Process of Fractional Two-Point Boundary Value Problem

2.1 Description about Fractional Boundary Value Problem

FBVP can be transformed to system of FIVP by following theorem from [10].

Theorem 2.1.1. *Multi-term Fractional Differential Equations with integer order term*

$$\begin{cases} D_a^{\alpha_n} y(t) = f(t, y(t), D_a^{\alpha_1} y(t), D_a^{\alpha_2} y(t), \dots, D_a^{\alpha_{n-1}} y(t)), \\ y^{(j)}(a) = y_a^{(j)}, \quad j = 0, 1, \dots, [\alpha_n], \end{cases} \quad (\text{II.2})$$

where $\alpha_n > \alpha_{n-1} > \dots > \alpha_1 > 0$, $\alpha_j - \alpha_{j-1} \leq 1$ for all $j = 2, 3, \dots, n$ and $0 < \alpha_1 < 1$. Then, we can define β_j , $j = 1, 2, \dots, n$,

$$\begin{cases} \beta_1 := \alpha_1, \\ \beta_j := \alpha_j - \alpha_{j-1}, \quad j = 2, 3, \dots, n, \end{cases} \quad (\text{II.3})$$

Then, this initial value problem (II.1) is equivalent to the system of equations

$$\begin{aligned} D_a^{\beta_1} y_0(t) &= y_1(t), \\ D_a^{\beta_2} y_1(t) &= y_2(t), \\ &\vdots \\ D_a^{\beta_{n-1}} y_{n-1}(t) &= y_n(t), \\ D_a^{\beta_n} y_n &= f(t, y_1, y_2, \dots, y_{n-2}, y_{n-1}), \end{aligned} \quad (\text{II.4a})$$

together with the initial conditions.

$$y_j(t_0) = \begin{cases} y^{(0)} & \text{if } j = 1, \\ y^{(l)} & \text{if } \alpha_{j-1} = l \in \mathbb{N}, \\ 0 & \text{else,} \end{cases} \quad (\text{II.4b})$$

in the following sense.

1. Whenever the function $y \in C^{[\alpha_n]}[a, b]$ is a solution of the multi-term equation II.2 with initial conditions, the vector-valued function $Y := (y_1, \dots, y_n)^T$ with

$$y_j(t) := \begin{cases} y(t) & \text{if } j = 1, \\ D_a^{\alpha_{j-1}} y(t) & \text{if } j = 2, \end{cases} \quad (\text{II.5})$$

is a solution of the multi-order fractional differential system (II.4a) with initial conditions (II.4b)

2. Whenever the vector-valued function $Y := (y_1, \dots, y_n)^T$ is a solution of the multi-order fractional differential system (II.4a) with initial condition (II.4b), the function $y := y_1$ is a solution of the multi-term equation II.2 with initial conditions.

There are many two-point boundary conditions. But in this research just consider Dirichlet and Robin boundary conditions.

Robin boundary condition is considered, similarly FBVP can be written as,

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y(t), D_a^{\alpha_1} y(t)), \\ a_1 y(t_0) + b_1 y'(t_0) = \gamma_1, \quad a_2 y(t_N) + b_2 y'(t_N) = \gamma_2 \end{cases} \quad (\text{II.6})$$

Then,

$$\begin{cases} D_a^{\alpha_1} y(t) = w(t), & y(t_0) = y_a = (\gamma_1 - b_1 s)/a_1 \\ D_a^{1-\alpha_1} w(t) = z(t), & w(t_0) = 0 \\ D_a^{\alpha_2-1} z(t) = f(t, y(t), w(t)), & z(t_0) = s. \end{cases} \quad (\text{II.7})$$

if b_1 and b_2 are zeros, that case is called “Dirichlet Boundary Condition”.

2.2 Numerical Methods

2.2.1 Fractional Two-Point Boundary Value Problem

Each equations of system (II.7) are FIVP. Then, solution is

$$\begin{cases} y(t) = y_0 + \frac{1}{\Gamma(\alpha_1)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha_1-1} w(\tau) d\tau, & y_0 = (\gamma_1 - b_1 s)/a_1 \\ w(t) = w_0 + \frac{1}{\Gamma(1-\alpha_1)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{-\alpha_1} z(\tau) d\tau, & w_0 = 0 \\ z(t) = z_0 + \frac{1}{\Gamma(\alpha_2-1)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha_2-2} f(\tau, y(\tau), w(\tau)) d\tau, & z_0 = s. \end{cases} \quad (\text{II.8})$$

Thus, numerical schemes for FIVP are needed. And in order to find unknown value $z_0 = s$ which induce boundary condition at the right end point $a_2 y(b) + b_2 y'(b) = \gamma_2$, shooting methods are used.

2.2.2 Numerical Methods for Fractional Initial Value Problem

Conventionally, “Predictor-Corrector Method” is adopted for numerical scheme. That is used to solve ordinary differential equation. In detail, It find an unknown function that satisfies a given differential equation. That process have two part, first, “prediction” part, it predicts points which fitted to the function value and derivative values from preceding set. Second, “corrector” part, it refines the previous approximated value which gained from “prediction” part, to obtain same point. Fractional differential equation can use “Predictor-Corrector Method” also, and it will be introduced bellow.

Before start, for convenience let's denote y_j as approximated value of $y(t_j)$ and $f_j \equiv f(t_j, y_j)$, $j = 1, \dots, N$. However, if $j = 0$ then, $f_0 = f(t_0, y(t_0))$. Because initial value $y(t_0)$ is given exactly. First, let the domain Ω to be

$$\Phi_N := \{ t_j \mid a = t_0 < \dots < t_j < \dots < t_n < t_{n+1} < \dots < t_N = b \}. \quad (\text{II.9})$$

for simplicity, step size is uniform, that means $t_{j+1} - t_j = h$, $j = 0, 1, \dots, N-1$. Thus (I.18) can be rewritten at time t_{n+1} as follows

$$y(t_{n+1}) = g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} f(\tau, y(\tau)) d\tau. \quad (\text{II.10})$$

where $g(t_{n+1}) = \sum_{k=0}^{[\alpha]} \frac{(t_{n+1}-a)^k}{k!} y_k$. For numerical approximations of $y(t)$, it need to interpolate the term $f(\tau, y(\tau))$, given sufficient continuity, over each interval $I_j = [t_j, t_{j+1}]$, $j =$

$0, 1, \dots, N-1$, by some types of interpolating polynomials. For this purpose, the Linear and Quadratic order Lagrange polynomials are often taken into consideration.

Thus, let's consider linear order Lagrange polynomials. On each interval $I_j = [t_j, t_{j+1}]$, Second order Lagrange interpolation of $f(\tau, y(\tau))$ is

$$f(\tau, y(\tau)) = L_j^1 f(\tau) + R_j(f(\tau)), \quad \tau \in [t_j, t_{j+1}] \quad (\text{II.11})$$

where

$$L_j^1 f(\tau) = \frac{t_{j+1} - \tau}{h} f(t_j, y(t_j)) - \frac{t_j - \tau}{h} f(t_{j+1}, y(t_{j+1})), \quad j = 0, 1, \dots, N,$$

and

$$\begin{aligned} R_j^1(f(\tau)) &= \frac{f''(\xi_j)}{2} (\tau - t_j)(\tau - t_{j+1}), \quad \xi_j \in (t_j, t_{j+1}). \\ y(t_{n+1}) &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (L_j^1 f(\tau) + R_j^1(f(\tau))) d\tau, \\ &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \left(B_{n+1}^{1,j} f(t_j, y(t_j)) + B_{n+1}^{2,j} f(t_{j+1}, y(t_{j+1})) \right) + T_{n+1}. \end{aligned} \quad (\text{II.12})$$

where

$$\begin{aligned} B_{n+1}^{1,j} &= \frac{1}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{j+1} - \tau) d\tau, \\ B_{n+1}^{2,j} &= -\frac{1}{h} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_j - \tau) d\tau, \\ T_{n+1} &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} R_j^1(f(\tau)) \tau. \end{aligned}$$

When $j = n$, it need $y(t_{n+1})$. But that term doesn't known. Thus, the $f(t_{n+1}, y(t_{n+1}))$ is approximated to f_{n+1}^P is needed. And that term must be 'predict', thus, it is called "predictor"

$$y_{n+1} = g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \left[\sum_{j=0}^{n-1} \left(B_{n+1}^{1,j} f_j + B_{n+1}^{2,j} f_{j+1} \right) + B_{n+1}^{1,n} f_n + B_{n+1}^{2,n} f_{n+1}^P \right], \quad (\text{II.13})$$

There are two types of 'predictor' for Second-order corrector. The initial is "Predictor-Evaluate Corrector-Evaluate Method(PECE)" type [14], and another is "Second-order High-Order Predictor Corrector Method(SHOM)" type [1].

"PECE" is the most used conventional method. It use rectangle rule to replace the right-hand side of (II.10).

$$y_{n+1}^P = g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n p_{n+1}^j f_j, \quad (\text{II.14})$$

where

$$p_{n+1}^j = \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} d\tau,$$

The convergence analysis shows that the error is expected to behave as

$$\max_{j=1, \dots, n} |y(t_j) - y_j| = O(h^q), \quad (\text{II.15})$$

where $q = \min(2, 1 + \alpha)$.

SHOM is another conventional one. This method have better convergence rate than PECE. Convergence rate of PECE depend on derivative order. But SHOM have fixed second-order convergence rate which does not depend on it.

$$y_{n+1}^P = g(t_{n+1}) + G(t_{n+1}) + b_{n+1}^1 f_{n-1} + b_{n+1}^2 f_n, \quad (\text{II.16})$$

where

$$\begin{aligned} G(t_{n+1}) &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n-1} \left(B_{n+1}^{1,j} f_j + B_{n+1}^{2,j+1} f_{j+1} \right), \\ b_{n+1}^1 &= \frac{1}{h} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_n - \tau) d\tau, \\ b_{n+1}^2 &= -\frac{1}{h} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{n-1} - \tau) d\tau. \end{aligned}$$

The Global error E_{n+1} of LHOM is

$$\max_{j=1, \dots, n} |y(t_j) - y_j| = O(h^2), \quad (\text{II.17})$$

(II.17) is guaranteed by the below theorem from [1].

Below global error theorem show convergence rate of SHOM.

Theorem 2.2.1. *Global Error of Second-order HOM Let define E_{n+1} is global error. Suppose $f(\cdot, y(\cdot)) \in C^2[a, b]$ and furthermore is Lipschitz continuous in the second argument in Theorem (1.2.6) then we have*

$$E_{n+1} = |y(t_{n+1}) - y_{n+1}| \leq O(h^2). \quad (\text{II.18})$$

Therefore, SHOM have better convergence rate than PECE.

Similarly, Quadratic Lagrange interpolation of $f(\tau, y(\tau))$ is defined as,

$$f(\tau, y(\tau)) = L_j^2 f(\tau) + R_j^2(f(\tau)), \quad t \in [t_j, t_{j+1}], \quad (\text{II.19})$$

where

$$\begin{aligned} L_j^2 f(\tau) &= f(t_{j-1}, y(t_{j-1})) Q_j^1(\tau) + f(t_j, y(t_j)) Q_j^2(\tau) + f(t_{j+1}, y(t_{j+1})) Q_j^3(\tau) \\ Q_j^1(\tau) &= \frac{(t_j - \tau)(t_{j+1} - \tau)}{(t_j - t_{j-1})(t_{j+1} - t_{j-1})}, \quad Q_j^2(\tau) = \frac{(t_{j-1} - \tau)(t_{j+1} - \tau)}{(t_{j-1} - t_j)(t_{j+1} - t_j)}, \quad Q_j^3(\tau) = \frac{(t_{j-1} - \tau)(t_j - \tau)}{(t_{j-1} - t_{j+1})(t_j - t_{j+1})}, \\ R_j^2(f(\tau)) &= \frac{f'''(\xi_j)}{6} (\tau - t_{j-1})(\tau - t_j)(\tau - t_{j+1}), \quad \xi_j \in (t_j, t_{j+1}), \quad j = 1, \dots, N. \end{aligned} \quad (\text{II.20})$$

But, $I_0 = [t_0, t_1]$ interval can not interpolate by Quadratic interpolation. So it will use a $t_{1/2}$ point. If exact value is known, just use it, or doesn't know it, approximate $y_{1/2} \equiv y_0$ or use startup scheme form [1]. Anyway, let's assume the value $y_{1/2}$ is known,

$$L_0^2 f(\tau) = Q_0^1(\tau) f(t_0, y(t_0)) + Q_0^2(\tau) f(t_{1/2}, y(t_{1/2})) + Q_0^3(\tau) f(t_1, y(t_1)), \quad (\text{II.21})$$

where

$$Q_0^1(\tau) = \frac{(t_{1/2} - \tau)(t_1 - \tau)}{(t_{1/2} - t_0)(t_1 - t_0)}, \quad Q_0^2(\tau) = \frac{(t_0 - \tau)(t_1 - \tau)}{(t_0 - t_{1/2})(t_1 - t_{1/2})}, \quad Q_0^3(\tau) = \frac{(t_0 - \tau)(t_{1/2} - \tau)}{(t_0 - t_1)(t_{1/2} - t_1)},$$

Thus, $y(t_{n+1})$ will be below form.

$$\begin{aligned} y(t_{n+1}) &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (L_j^2 f(\tau) + R_j^2(f(\tau))) d\tau, \\ &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \left[A_{n+1}^{1,0} f(t_0, y(t_0)) + A_{n+1}^{2,0} f(t_{1/2}, y(t_{1/2})) + A_{n+1}^{3,0} f(t_1, y(t_1)) \right. \\ &\quad \left. + \sum_{j=1}^n \left(A_{n+1}^{1,j} f(t_{j-1}, y(t_{j-1})) + A_{n+1}^{2,j} f(t_j, y(t_j)) + A_{n+1}^{3,j} f(t_{j+1}, y(t_{j+1})) \right) \right] \\ &\quad + T_{n+1}. \end{aligned} \tag{II.22}$$

where

$$\begin{aligned} A_{n+1}^{1,0} &= \frac{2}{h^2} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{1/2} - \tau)(t_1 - \tau) d\tau, \\ A_{n+1}^{2,0} &= -\frac{4}{h^2} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_0 - \tau)(t_1 - \tau) d\tau, \\ A_{n+1}^{3,0} &= \frac{2}{h^2} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_0 - \tau)(t_{1/2} - \tau) d\tau, \\ A_{n+1}^{1,j} &= \frac{1}{2h^2} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_j - \tau)(t_{j+1} - \tau) d\tau, \\ A_{n+1}^{2,j} &= -\frac{1}{h^2} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{j-1} - \tau)(t_{j+1} - \tau) d\tau, \\ A_{n+1}^{3,j} &= \frac{1}{2h^2} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{j-1} - \tau)(t_j - \tau) d\tau, \\ T_{n+1} &= \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} R_j^2(f(\tau)) d\tau. \end{aligned}$$

Similarly, When $j = n$, it need $y(t_{n+1})$. But that term doesn't known. Thus, the $f(t_{n+1}, y(t_{n+1}))$ is approximated to f_{n+1}^P is needed like Linear corrector. Thus, also that term is predicted, and it will be called "predictor".

if $n = 1$ case,

$$y_1 = g(t_1) + \frac{1}{\Gamma(\alpha)} \left(A_{n+1}^{1,0} f_0 + A_{n+1}^{2,0} f_{1/2} + A_{n+1}^{3,0} f_1^P \right), \tag{II.23}$$

and other case, i.e $n \neq 1$,

$$\begin{aligned} y_{n+1} &= g(t_{n+1}) + \frac{1}{\Gamma(\alpha)} \left[A_{n+1}^{1,0} f_0 + A_{n+1}^{2,0} f_{1/2} + A_{n+1}^{3,0} f_1 \right. \\ &\quad \left. + \sum_{j=1}^{n-1} \left(A_{n+1}^{1,j} f_{j-1} + A_{n+1}^{2,j} f_j + A_{n+1}^{3,j} f_{j+1} \right) + A_{n+1}^{1,n} f_{n-1} + A_{n+1}^{2,n} f_n + A_{n+1}^{3,n} f_{n+1}^P \right]. \end{aligned} \tag{II.24}$$

Predictor of Third-order High-Order Predictor Corrector Method(THOM)" is induced as bellow.

$$y^P(t_{n+1}) = g(t_{n+1}) + G(t_{n+1}) + a_{n+1}^1 f_{n-2} + a_{n+1}^2 f_{n-1} + a_{n+1}^3 f_n, \quad (\text{II.25})$$

where

$$G(t_{n+1}) = \frac{1}{\Gamma(\alpha)} \left[A_{n+1}^{1,0} f_0 + A_{n+1}^{2,0} f_{1/2} + A_{n+1}^{3,0} f_1 + \sum_{j=0}^{n-1} \left(A_{n+1}^{1,j} f_{j-1} + A_{n+1}^{2,j} f_j + A_{n+1}^{3,j} f_{j+1} \right) \right], \quad (\text{II.26})$$

$$a_{n+1}^1 = \frac{1}{2h^2} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{n-1} - \tau)(t_n - \tau) d\tau,$$

$$a_{n+1}^2 = -\frac{1}{h^2} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{n-2} - \tau)(t_n - \tau) d\tau,$$

$$a_{n+1}^3 = \frac{1}{2h^2} \int_{t_n}^{t_{n+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{n-2} - \tau)(t_{n-1} - \tau) d\tau.$$

The Global error E_{n+1} of THOM is

$$\max_{j=1, \dots, n} |y(t_j) - y_j| = O(h^3). \quad (\text{II.27})$$

Either (II.27) is guaranteed by the below theorem from [1].

Theorem 2.2.2. *Global Error of Third-order HOM Suppose $f(\cdot, y(\cdot)) \in C^3[a, b]$ and is Lipschitz continuous in the second argument in Theorem (1.2.6) then we have*

$$E_{n+1} \leq O(h^3) \quad (\text{II.28})$$

given $E_1, E_2 \leq O(h^3)$ and $E_{1/2} \leq O(h^{3-\alpha})$, $(0 < \alpha < 1)$, $O(h^2)$, $(\alpha > 1)$.

2.2.3 Shooting methods

To approximate $y'(t_0) = s$, using shooting method. First, let's start to define $y(s) := y(t, s)|_{t=b}$. And error function $F(s)$ is defined by,

$$F(s) = a_2 y(s) + b_2 y'(s) - \gamma_2. \quad (\text{II.29})$$

Then, to find zero of error function $F(s)$, Newton's and Halley's methods are used. For convenience of writing, let's notate two variable function $f(t, s)$,

$$\frac{\partial f(t, s)}{\partial s} = f_s, \quad \frac{\partial^2 f(t)}{\partial s^2} = f_{ss}. \quad (\text{II.30})$$

Newton's Method with Robin Boundary Condition

The general Newton's Method is

$$s_{k+1} = s_k - \frac{F(s_k)}{F'(s_k)}, \quad k = 0, 1, 2, \dots, \quad (\text{II.31})$$

and it need to get

$$F'(s) = y_s(t, s)|_{t=b}. \quad (\text{II.32})$$

Therefore the operator $\frac{\partial}{\partial s}$ is applied to (II.7).

$$\begin{cases} D_a^{\alpha_1} y_s(t) = w_s(t), & y_s(t_0) = (\gamma_1 - b_1)/a_1 \\ D_a^{1-\alpha_1} w_s(t) = z_s(t), & w_s(t_0) = 0 \\ D_a^{\alpha_2-1} z_s(t) = f_s(t, y(t), w(t)), & z_s(t_0) = 1 \end{cases} \quad (\text{II.33})$$

t and s are independent, so $f_s(t, y(t), w(t))$ can be written as,

$$f_s(t, y(t), w(t)) = f_y \cdot y_s(t) + f_w \cdot w_s(t), \quad (\text{II.34})$$

Let, define

$$y_s = \hat{y}, \quad w_s = \hat{w}, \quad z_s = \hat{z}. \quad (\text{II.35})$$

Then, (II.33) can be rewritten as

$$\begin{cases} D_a^{\alpha_1} \hat{y}(t) = \hat{w}(t), & \hat{y}(t_0) = (\gamma_1 - b_1)/a_1 \\ D_a^{1-\alpha_1} \hat{w}(t) = \hat{z}(t), & \hat{w}(t_0) = 0 \\ D_a^{\alpha_2-1} \hat{z}(t) = f_y \cdot \hat{y}(t) + f_w \cdot \hat{w}(t), & \hat{z}(t_0) = 1 \end{cases} \quad (\text{II.36})$$

Thus, s can be approximated by solving above system and it will be used as initial condition to solve the system (??). Also, approximated value $y(s, b)$ is gotten, then $F(s)$ can be calculated. By that process will show amount of error, and if that is bigger, then Newton's Method will be iterated enough when $F(s)$ becomes almost zero.

Halley's Method with Robin Boundary Condition

The general Halley's Method is

$$s_{k+1} = s_k - \frac{2F(s_k)F'(s_k)}{2F'^2(s_k) - F(s_k)F''(s_k)}, \quad k = 0, 1, 2, \dots, \quad (\text{II.37})$$

Similarly, to get $F''(s_k)$, operator $\frac{\partial^2}{\partial s^2}$ is applied to (II.7).

$$\begin{cases} D_a^{\alpha_1} y_{ss}(t) = w_{ss}(t), & y_{ss}(t_0) = 0 \\ D_a^{1-\alpha_1} w_{ss}(t) = z_{ss}(t), & w_{ss}(t_0) = 0 \\ D_a^{\alpha_2-1} z_{ss}(t) = f_{ss}(t, y(t), w(t)), & z_{ss}(t_0) = 0 \end{cases} \quad (\text{II.38})$$

t and s are independent, so $f_{ss}(t, y(t), w(t))$ can be written as,

$$f_y \cdot y_{ss}(t) + f_w \cdot w_{ss}(t) + f_{yy} \cdot y_s(t)^2 + f_{ww} \cdot w_s(t)^2 + f_{wy} \cdot w_s(t)y_s(t), \quad (\text{II.39})$$

Let, define

$$y_{ss} = \tilde{y}, \quad w_{ss} = \tilde{w}, \quad z_{ss} = \tilde{z}. \quad (\text{II.40})$$

$$\begin{cases} D_a^{\alpha_1} \tilde{y}(t) = \tilde{w}(t), & \tilde{y}(t_0) = 0 \\ D_a^{1-\alpha_1} \tilde{w}(t) = \tilde{z}(t), & \tilde{w}(t_0) = 0 \\ D_a^{\alpha_2-1} \tilde{z}(t) = f_y \cdot \tilde{y}(t) + f_w \cdot \tilde{w}(t) + f_{yy} \cdot \hat{y}(t)^2 + f_{ww} \cdot \hat{w}(t)^2 + f_{wy} \cdot \hat{w}(t)\hat{y}(t), & \tilde{z}(t_0) = 0 \end{cases} \quad (\text{II.41})$$

and y'_s can be founded like below.

$$z_s = D_a^{1-\alpha_1} D_a^{\alpha_1} y_s(t) = D^1 y_s(t) = y'_s(t). \quad (\text{II.42})$$

In general, y , w and z are solved by PECE, SHOM or THOM. Conventional methods need predictor for y , w and z for each iteration. But this research will suggest new method that comes from specific structure of nonlinear system. When y_{n+1} term is calculated then it must need w_{n+1} . So w_{n+1} term is calculated then, z_{n+1} must be needed also. However z_{n+1} needs y_{n+1} and w_{n+1} term. Therefore a system of FIVP have special iterative structure. Thus, first, predictor y_{n+1}^P and w_{n+1}^P are found by PECE or SHOM or THOM(in this paper SHOM and THOM is used), then z_{n+1} can be calculated. Second, discovered z_{n+1} is used as predictor to find w_{n+1} , and finally y_{n+1} can be found by using w_{n+1} as predictor. This method is called “Hybrid Method”(HM). Also above structure can be iterated, because if one circular is accomplished then, there are y_{n+1} and w_{n+1} . Thus, above process can be iterated. But it must stop sometime, so it will use below stopping criterion.

$$z_i^c = z_0 + J_a^{\alpha_2-1} f_i \quad (\text{II.43})$$

whose $z_0 = y'_0$ is initial value of z , then

$$E^i = \frac{z_i^c - z_0}{J_a^{\alpha_2-1} f_i}, \quad E^i \approx 1, \Rightarrow \frac{|E^i - E^{i+1}|}{E^i} \ll Tol, \quad i \leq I_{\max}. \quad (\text{II.44})$$

This scheme will be called “Iterated Hybrid Method”(IHM). Figure.2 shows a process of HM and IHM. And similar to SHOM and THOM there are SHM, THM, SIHM and TIHM according to kind of interpolation function.

2.3 Numerical Results

This section will show the accuracy and efficiency of new methods. There are two kinds of table. First one is approximated error(II.45). It will illustrate shooting methods are how well it work, Second table is convergence rate of maximum error. It will show accuracy of methods. h is step size and N is number of grid on closed interval $[0, 1]$. s is assumed start initial value for $y'(t_0)$. Approximated error choose $h = 0.01$, and Convergence rate chooses $s = 0.2$. For convenience let's write it as shown in the table below.

$$|F(s)| = |a_2 y(s) + b_2 y'(s) - \gamma_2| \quad \text{Approximated error} \quad (\text{II.45})$$

IHM have 5 maximum iteration number and tolerance is 10^{-6} .

Example 2.3.1. *Polynomial type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = \frac{\Gamma(5)}{\Gamma(5-\alpha_2)} t^{4-\alpha_2} + \frac{\Gamma(5)}{\Gamma(5-\alpha_1)} t^{4-\alpha_1} - t^8 + y^2 - D_0^{\alpha_1} y(t), \\ a_1 y(0) + b_1 y'(0) = \gamma_1, \quad a_2 y(1) + b_2 y'(1) = \gamma_2. \end{cases} \quad (\text{II.46})$$

$$\begin{cases}
 D_a^{\alpha_1} \boxed{y_{n+1}} = \boxed{w_{n+1}}, & y(t_0) = y_a \\
 D_a^{1-\alpha_1} \boxed{w_{n+1}} = \boxed{z_{n+1}}, & w(t_0) = w_a \\
 D_a^{\alpha_2-1} \boxed{z_{n+1}} = f(t_{n+1}, \boxed{y_{n+1}}, \boxed{w_{n+1}}), & z(t_0) = s
 \end{cases}$$

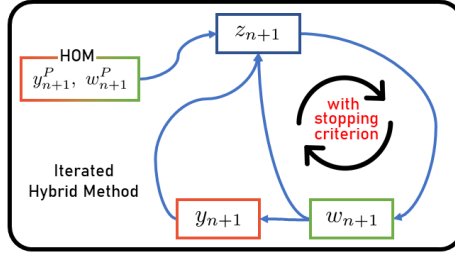
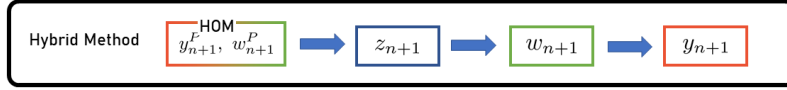


Figure 2: Structure of HM and IHM

Where $\alpha_2 = 1.7$, $\alpha_1 = 0.4$. exact solution is

$$y(t) = y^4. \quad (\text{II.47})$$

First, $a_1 = 1$, $b_1 = 0$, $a_2 = 1$, $b_2 = 0$. That case is called ‘Dirichlet Boundary Condition’.

original	notation
PECE method with Newton's method	NPECE
PECE method with Halley's method	HPECE
SHOM with Newton's method	NHOM
THOM with Halley's method	HHOM
SHM with Newton's method	NHM
THM with Halley's method	HHM
SIHMwith Newton's method	NIHM
TIHM with Halley's method	HIHM

Notations

	NPECE					HPECE				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.31906	0.66534	1.03376	1.42592	1.84355	0.31906	0.66534	1.03376	1.42592	1.84355
2	0.00921	0.037	0.08281	0.14629	0.22727	0.03334	0.15103	0.38041	0.75613	1.31981
3	9.78E-06	0.00014	0.00067	0.00203	0.00478	0.00036	0.00712	0.04389	0.16613	0.47578
4	1.69E-09	2.55E-08	1.57E-07	7.46E-07	3.03E-06	1.01E-07	1.72E-05	0.00061	0.00859	0.0679
5	2.89E-13	4.37E-12	2.70E-11	1.28E-10	5.21E-10	1.73E-11	3.05E-09	2.24E-07	2.48E-05	0.00146
6	2.22E-16	4.44E-16	4.44E-15	2.22E-14	8.97E-14	2.89E-15	5.22E-13	3.85E-11	4.45E-09	9.23E-07
7	2.22E-16	0	2.22E-16	2.22E-16	2.22E-16	2.22E-16	4.44E-16	6.55E-15	7.63E-13	1.58E-10
8	2.22E-16	0	0	0	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16	2.71E-14
9	2.22E-16	0	0	0	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16	2.22E-16
10	2.22E-16	0	0	0	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16	0

Table 1: Approximated error of PECE with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

	NPECE		HPECE	
N	error	roc	error	roc
10	0.03929	-	0.03929	-
20	0.01501	1.38817	0.01501	1.38817
40	0.00558	1.42644	0.00558	1.42644
80	0.00205	1.44705	0.00205	1.44705
160	7.47E-04	1.45535	7.47E-04	1.45535
320	2.72E-04	1.45765	2.72E-04	1.45765

Table 2: Convergence rate of PECE methods with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

NHOM						HHOM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.32715	0.67498	1.04536	1.43997	1.86061	0.32682	0.67466	1.04505	1.43965	1.86028
2	0.00891	0.03629	0.0816	0.1445	0.22488	2.50E-05	0.00042	0.0018	0.00456	0.00916
3	9.12E-06	5.73E-05	0.00047	0.00167	0.00418	8.08E-09	1.35E-07	5.79E-07	1.47E-06	2.95E-06
4	1.71E-08	1.07E-07	8.69E-07	2.86E-06	6.18E-06	2.61E-12	4.36E-11	1.87E-10	4.75E-10	9.53E-10
5	3.22E-11	2.02E-10	1.63E-09	5.38E-09	1.16E-08	8.88E-16	1.40E-14	6.05E-14	1.53E-13	3.08E-13
6	6.04E-14	3.79E-13	3.07E-12	1.01E-11	2.18E-11	0	0	0	2.22E-16	4.44E-16
7	1.11E-16	6.66E-16	5.77E-15	1.89E-14	4.09E-14	0	0	0	2.22E-16	2.22E-16
8	1.11E-16	1.11E-16	1.11E-16	1.11E-16	1.11E-16	0	0	0	2.22E-16	0
9	2.22E-16	0	0	2.22E-16	2.22E-16	0	0	0	2.22E-16	0
10	1.11E-16	0	0	1.11E-16	1.11E-16	0	0	0	2.22E-16	0

Table 3: Approximated error of HOMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

N	NHOM		HHOM	
	error	roc	error	roc
10	0.005	-	1.07E-03	-
20	0.00143	1.80632	8.36E-05	3.67563
40	0.00037	1.96494	5.81E-06	3.8473
80	9.18E-05	1.99605	3.84E-07	3.92012
160	2.29E-05	2.00064	2.49E-08	3.94409
320	5.74E-06	2.0003	1.67E-09	3.89623

Table 4: Convergence rate of HOMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

NHM						HHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.32715	0.67499	1.04537	1.43998	1.86063	0.32682	0.67466	1.04505	1.43965	1.86028
2	0.00968	0.03818	0.08496	0.14968	0.23225	0.00011	0.00075	0.00241	0.00556	0.01064
3	9.17E-06	1.41E-04	0.00069	0.00211	0.00499	8.98E-12	1.97E-10	1.75E-09	9.13E-09	3.44E-08
4	3.42E-11	2.35E-09	4.86E-08	4.42E-07	2.44E-06	2.22E-16	2.22E-16	0	1.11E-15	1.55E-15
5	2.22E-16	6.66E-15	1.38E-13	1.27E-12	7.50E-12	2.22E-16	2.22E-16	0	4.44E-16	2.22E-16
6	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16
7	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16
8	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16
9	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16
10	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	2.22E-16	0	2.22E-16	2.22E-16

Table 5: Approximated error of HMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

	NHM		HHM	
N	error	roc	error	roc
10	0.00513	-	9.36E-04	-
20	0.00141	1.86003	7.40E-05	3.66082
40	0.00036	1.96275	5.16E-06	3.84305
80	9.14E-05	1.98976	3.41E-07	3.91796
160	2.29E-05	1.99673	2.20E-08	3.95315
320	5.73E-06	1.99838	1.73E-09	3.6722

Table 6: Convergence rate of HMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

	NIHM					HIHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.32718	0.67502	1.04542	1.44005	1.86072	0.32682	0.67466	1.04505	1.43965	1.86028
2	0.00968	0.03819	0.08497	0.1497	0.23228	0.00011	0.00075	0.00241	0.00555	0.01064
3	9.14E-06	1.41E-04	0.00069	0.00211	0.00499	3.50E-12	1.59E-10	1.63E-09	8.86E-09	3.39E-08
4	8.98E-12	1.85E-09	4.65E-08	4.36E-07	2.43E-06	0	0	2.22E-16	2.22E-16	2.22E-16
5	2.22E-16	2.22E-16	4.66E-15	5.71E-14	7.91E-13	0	0	2.22E-16	2.22E-16	2.22E-16
6	0	0	5.55E-16	2.22E-16	0	0	0	2.22E-16	2.22E-16	2.22E-16
7	0	0	0	0	0	0	0	2.22E-16	2.22E-16	2.22E-16
8	0	0	0	0	0	0	0	2.22E-16	2.22E-16	2.22E-16
9	0	0	0	0	0	0	0	2.22E-16	2.22E-16	2.22E-16
10	0	0	0	0	0	0	0	2.22E-16	2.22E-16	2.22E-16

Table 7: Approximated error of IHMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

	NIHM		HIHM	
N	error	roc	error	roc
10	0.00563	-	0.00026	-
20	0.00143	1.97608	1.89E-05	3.76896
40	0.00036	1.98682	1.30E-06	3.86003
80	9.07E-05	1.99257	9.26E-08	3.81318
160	2.28E-05	1.99343	1.46E-08	2.6653
320	5.71E-06	1.99582	2.10E-09	2.79573

Table 8: Convergence rate of IHMs with Polynomial type example and Dirichlet Boundary condition in example 2.3.1.

Second, $a_1 = 1$, $b_1 = 1$, $a_2 = 1$, $b_2 = 1$. That case is called ‘Robin Boundary Condition’.

	NPECE					HPECE				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.58894	1.28888	2.08715	2.98674	3.9931	0.58894	1.28888	2.08715	2.98674	3.9931
2	0.04035	0.14513	0.29984	0.50086	0.74941	0.07202	0.25004	0.50828	0.84308	1.25752
3	2.65E-04	0.00315	0.01219	0.03049	0.06063	0.00155	0.01613	0.05621	0.12933	0.24089
4	5.07E-08	2.08E-06	2.59E-05	1.53E-04	5.86E-04	9.81E-07	8.32E-05	0.00096	0.00474	0.01507
5	7.51E-12	3.09E-10	3.95E-09	2.66E-08	1.43E-07	1.46E-10	1.45E-08	4.30E-07	7.75E-06	7.29E-05
6	8.88E-16	4.62E-14	5.84E-13	3.93E-12	2.12E-11	2.13E-14	2.15E-12	6.37E-11	1.17E-09	1.25E-08
7	0	1.78E-15	8.88E-16	8.88E-16	2.66E-15	0	0	9.77E-15	1.74E-13	1.85E-12
8	0	8.88E-16	0	8.88E-16	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0

Table 9: Approximated error of PECE with Polynomial type example and Robin Boundary condition in example 2.3.1.

	NPECE		HPECE	
N	error	roc	error	roc
10	0.11617	-	0.11617	-
20	0.04685	1.31023	0.04685	1.31023
40	0.0177	1.40446	0.0177	1.40446
80	0.00648	1.45003	0.00648	1.45003
160	2.34E-03	1.47055	2.34E-03	1.47055
320	8.42E-04	1.47363	8.42E-04	1.47363

Table 10: Convergence rate of PECE methods with Polynomial type example and Robin Boundary condition in example 2.3.1.

	NHOM					HHOM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.60401	1.30554	2.10577	3.00776	4.01705	0.60319	1.3047	2.10484	3.00664	4.01565
2	0.04223	0.14833	0.30385	0.50516	0.75349	0.05433	0.19029	0.3923	0.65925	0.99521
3	2.74E-04	3.25E-03	0.01249	0.03101	0.06134	0.0006	0.00684	0.02572	0.06314	0.12416
4	7.31E-08	7.33E-07	2.17E-05	1.46E-04	5.78E-04	9.11E-08	8.24E-06	0.00013	0.00081	0.00304
5	2.28E-11	2.29E-10	6.71E-09	4.22E-08	1.25E-07	2.58E-11	2.32E-09	3.41E-08	8.56E-08	1.15E-06
6	7.11E-15	7.19E-14	2.10E-12	1.32E-11	3.91E-11	6.22E-15	6.55E-13	9.64E-12	2.42E-11	3.26E-10
7	0	0	8.88E-16	4.44E-15	1.07E-14	8.88E-16	0	2.66E-15	6.22E-15	9.15E-14
8	0	0	0	0	1.78E-15	8.88E-16	0	8.88E-16	0	8.88E-16
9	0	0	0	0	0	8.88E-16	0	8.88E-16	0	8.88E-16
10	0	0	0	0	0	8.88E-16	0	8.88E-16	0	8.88E-16

Table 11: Approximated error of HOMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

	NHOM		HHOM	
N	error	roc	error	roc
10	0.01173	-	1.02E-03	-
20	0.00563	1.05861	1.83E-04	2.48635
40	0.00178	1.66467	3.63E-05	2.33187
80	4.92E-04	1.85351	5.47E-06	2.73014
160	1.29E-04	1.93106	7.51E-07	2.8661
320	3.30E-05	1.96608	9.88E-08	2.92502

Table 12: Convergence rate of HOMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

	NHM					HHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.60402	1.30555	2.10578	3.00777	4.01705	0.6032	1.30473	2.10489	3.00674	4.0158
2	0.0423	0.14851	0.30441	0.50657	0.75634	0.05452	0.19077	0.39317	0.66064	0.99726
3	2.87E-04	3.29E-03	0.01258	0.03123	0.06181	0.00062	0.00693	0.02594	0.06358	0.12492
4	1.39E-08	1.79E-06	2.59E-05	1.58E-04	6.04E-04	8.50E-08	1.04E-05	0.00014	0.00084	0.00311
5	1.60E-14	2.55E-12	1.40E-10	4.27E-09	6.09E-08	0	2.38E-11	4.55E-09	1.55E-07	2.11E-06
6	8.88E-16	0	0	6.22E-15	6.93E-14	0	8.88E-16	0	2.66E-15	9.57E-13
7	0	0	0	8.88E-16	8.88E-16	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0

Table 13: Approximated error of HMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

	NHM		HHM	
N	error	roc	error	roc
10	0.01695	-	6.52E-04	-
20	0.00626	1.43727	2.07E-04	1.6579
40	0.00185	1.76021	3.78E-05	2.4509
80	4.99E-04	1.88733	5.56E-06	2.76481
160	1.30E-04	1.94437	7.55E-07	2.87921
320	3.31E-05	1.97151	9.91E-08	2.93034

Table 14: Convergence rate of HMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

	NIHM					TIHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.60407	1.30561	2.10585	3.00786	4.01718	0.60319	1.30472	2.10488	3.00673	4.01578
2	0.04231	0.14852	0.30443	0.50659	0.75637	0.05452	0.19077	0.39317	0.66064	0.99725
3	2.87E-04	3.29E-03	0.01259	0.03123	0.06181	0.00062	0.00693	0.02594	0.06358	0.12492
4	1.32E-08	1.79E-06	2.59E-05	1.58E-04	6.04E-04	8.50E-08	1.04E-05	0.00014	0.00084	0.00311
5	0	5.52E-13	1.11E-10	3.74E-09	5.96E-08	2.66E-15	2.39E-11	4.55E-09	1.55E-07	2.11E-06
6	0	0	8.88E-16	0	1.78E-15	0	1.78E-15	8.88E-16	4.44E-15	9.75E-13
7	0	0	0	0	0	0	0	8.88E-16	0	8.88E-16
8	0	0	0	0	0	0	0	8.88E-16	0	0
9	0	0	0	0	0	0	0	8.88E-16	0	0
10	0	0	0	0	0	0	0	8.88E-16	0	0

Table 15: Approximated error of IHMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

	NIHM		TIHM	
N	error	roc	error	roc
10	0.03342	-	2.26E-03	-
20	0.00834	2.00179	3.30E-04	2.77993
40	0.0021	1.98766	4.52E-05	2.86784
80	5.30E-04	1.98821	6.00E-06	2.91218
160	1.33E-04	1.99021	7.82E-07	2.94025
320	3.35E-05	1.99305	1.01E-07	2.95705

Table 16: Convergence rate of IHMs with Polynomial type example and Robin Boundary condition in example 2.3.1.

Example 2.3.2. *Exponential type example.*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) + y^2 + t D_0^{\alpha_1} y(t), \\ a_1 y(0) + b_1 y'(0) = \gamma_1, \quad a_2 y(1) + b_2 y'(1) = \gamma_2. \end{cases} \quad (\text{II.48})$$

whose $\alpha_2 = 1.7$, $\alpha_1 = 0.4$,

$$\begin{aligned} F(t) = & t^{2-\alpha_2} E_{1,3-\alpha_2}(t) - \left(\frac{\Gamma(3)}{2\Gamma(3-\alpha_2)} t^{2-\alpha_2} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)} t^{3-\alpha_2} \right) \\ & - \left\{ e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right) \right\}^2 \\ & - t \left\{ t^{1-\alpha_1} E_{1,2-\alpha_1}(t) - \left(\frac{\Gamma(2)}{\Gamma(2-\alpha_1)} t^{1-\alpha_1} + \frac{\Gamma(3)}{2\Gamma(3-\alpha_1)} t^{2-\alpha_1} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_1)} t^{3-\alpha_1} \right) \right\} \end{aligned}$$

and exact solution is

$$y(t) = e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right). \quad (\text{II.49})$$

First, $a_1 = 1$, $b_1 = 0$, $a_2 = 1$, $b_2 = 0$. That case is called ‘Dirichlet Boundary Condition’.

	NPECE					HPECE				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.24714	0.50854	0.78474	1.07665	1.38528	0.24714	0.50854	0.78474	1.07665	1.38528
2	0.00646	0.02556	0.0569	0.10022	0.15538	0.02653	0.11799	0.29617	0.5898	1.03555
3	4.78E-06	7.37E-05	0.00036	0.00111	0.00262	0.00029	0.00566	0.03457	0.13035	0.37355
4	4.57E-11	1.28E-09	1.81E-08	1.49E-07	8.01E-07	3.78E-08	1.33E-05	0.00049	0.0069	0.05426
5	4.02E-16	1.16E-14	1.63E-13	1.35E-12	7.30E-12	3.42E-13	1.94E-10	1.06E-07	1.98E-05	0.00121
6	1.39E-17	0	1.39E-17	2.78E-17	6.25E-17	0	1.75E-15	9.57E-13	3.41E-10	6.20E-07
7	0	0	0	0	1.39E-17	0	0	0	3.07E-15	5.75E-12
8	0	0	0	0	0	0	0	0	0	4.16E-17
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0

Table 17: Approximated error of PECE with Exponential type example and Dirichlet Boundary condition.

	NPECE		HPECE	
N	error	roc	error	roc
10	0.00253	-	0.00253	-
20	0.00098	1.36571	0.00098	1.36571
40	0.00037	1.4066	0.00037	1.4066
80	0.00014	1.42948	0.00014	1.42948
160	5.07E-05	1.43951	5.07E-05	1.43951
320	1.86E-05	1.44369	1.86E-05	1.44369

Table 18: Convergence rate of PECE methods with Exponential type example and Dirichlet Boundary condition

	NHOM					HHOM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.24779	0.50964	0.78647	1.07923	1.38896	0.24777	0.50962	0.78645	1.0792	1.38892
2	0.00639	0.02528	0.05632	0.09925	0.15394	4.51E-05	0.00038	0.0013	0.00308	0.006
3	3.92E-06	6.85E-05	0.00034	0.00106	0.00252	7.63E-10	6.49E-09	2.20E-08	5.21E-08	1.01E-07
4	3.88E-10	6.28E-09	2.09E-08	2.07E-08	4.62E-07	1.29E-14	1.10E-13	3.73E-13	8.82E-13	1.71E-12
5	3.85E-14	6.24E-13	2.08E-12	2.06E-12	4.58E-11	6.94E-18	6.94E-18	6.94E-18	1.39E-17	1.39E-17
6	6.94E-18	6.94E-17	2.08E-16	2.15E-16	4.55E-15	0	0	0	6.94E-18	6.94E-18
7	6.94E-18	0	0	6.94E-18	6.94E-18	0	0	0	0	0
8	0	0	0	6.94E-18	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0

Table 19: Approximated error of HOMs with Exponential type example and Dirichlet Boundary condition.

	NHOM		HHOM	
N	error	roc	error	roc
10	0.00029	-	5.00E-05	-
20	9.66E-05	1.59343	4.64E-06	3.43146
40	2.67E-05	1.85252	4.52E-07	3.35999
80	6.99E-06	1.9361	4.92E-08	3.19831
160	1.78E-06	1.96891	5.70E-09	3.11099
320	4.51E-07	1.98378	6.82E-10	3.06297

Table 20: Convergence rate of HOMs with Exponential type example and Dirichlet Boundary condition.

	NHM					HHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.24779	0.50964	0.78647	1.07923	1.38896	0.24777	0.50962	0.78645	1.0792	1.38892
2	0.00653	0.02582	0.05747	0.10126	0.15704	6.47E-05	0.00046	0.00147	0.00339	0.0065
3	4.86E-06	7.55E-05	0.00037	0.00114	0.00269	1.27E-12	5.42E-11	5.57E-10	3.03E-09	1.16E-08
4	3.48E-12	6.62E-10	1.57E-08	1.47E-07	8.26E-07	6.94E-18	6.94E-18	0	0	4.16E-17
5	0	9.71E-17	2.58E-15	2.62E-14	2.11E-13	0	0	0	0	0
6	0	0	1.39E-17	1.39E-17	0	0	0	0	0	0
7	0	0	1.39E-17	1.39E-17	0	0	0	0	0	0
8	0	0	1.39E-17	1.39E-17	0	0	0	0	0	0
9	0	0	1.39E-17	1.39E-17	0	0	0	0	0	0
10	0	0	1.39E-17	1.39E-17	0	0	0	0	0	0

Table 21: Approximated error of HMs with Exponential type example and Dirichlet Boundary condition.

	NHM		HHM	
N	error	roc	error	roc
10	0.00032	-	4.95E-05	-
20	9.97E-05	1.7012	4.31E-06	3.5222
40	2.70E-05	1.88283	4.36E-07	3.30421
80	7.02E-06	1.94631	4.84E-08	3.1713
160	1.79E-06	1.97271	5.66E-09	3.0981
320	4.52E-07	1.98524	6.79E-10	3.05715

Table 22: Convergence rate of HMs with Exponential type example and Dirichlet Boundary condition.

	NIHM					HIHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.24779	0.50965	0.78648	1.07924	1.38898	0.24777	0.50962	0.78645	1.0792	1.38892
2	0.00653	0.02582	0.05748	0.10127	0.15705	6.47E-05	0.00046	0.00147	0.00339	0.0065
3	4.86E-06	7.55E-05	0.00037	0.00114	0.00269	1.06E-12	5.27E-11	5.52E-10	3.01E-09	1.15E-08
4	2.64E-12	6.50E-10	1.57E-08	1.47E-07	8.25E-07	0	0	0	1.39E-17	6.94E-18
5	1.39E-17	0	1.11E-16	3.23E-15	8.20E-14	0	0	0	0	0
6	1.39E-17	0	1.39E-17	0	1.39E-17	0	0	0	0	0
7	1.39E-17	0	1.39E-17	0	2.78E-17	0	0	0	0	0
8	1.39E-17	0	1.39E-17	0	1.39E-17	0	0	0	0	0
9	1.39E-17	0	1.39E-17	0	1.39E-17	0	0	0	0	0
10	1.39E-17	0	1.39E-17	0	2.78E-17	0	0	0	0	0

Table 23: Approximated error of IHMs with Exponential type example and Dirichlet Boundary condition.

	NIHM		HIHM	
N	error	roc	error	roc
10	0.00043	-	2.75E-05	-
20	1.12E-04	1.95258	3.09E-06	3.15453
40	2.84E-05	1.97845	3.64E-07	3.08536
80	7.17E-06	1.98577	4.40E-08	3.04977
160	1.81E-06	1.98968	5.38E-09	3.03233
320	4.54E-07	1.99294	6.62E-10	3.02209

Table 24: Convergence rate of IHMs with Exponential type example and Dirichlet Boundary condition.

Second, $a_1 = 1$, $b_1 = 1$, $a_2 = 1$, $b_2 = 1$. That case is called ‘Robin Boundary Condition’.

	NPECE					HPECE				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.41336	0.88951	1.42505	2.01898	2.67196	0.41336	0.88951	1.42505	2.01898	2.67196
2	0.02735	0.09401	0.18881	0.30849	0.45321	0.03893	0.1323	0.26772	0.44499	0.66775
3	0.00017	0.00186	0.00686	0.01653	0.03194	0.0005	0.00518	0.01851	0.044	0.08479
4	7.27E-09	8.10E-07	1.09E-05	6.24E-05	0.00023	8.83E-08	9.25E-06	0.00012	0.00063	0.00224
5	2.83E-14	3.30E-12	6.98E-11	1.15E-09	1.31E-08	3.46E-13	6.56E-11	5.10E-09	1.42E-07	1.75E-06
6	0	0	2.78E-16	4.44E-15	5.08E-14	0	2.78E-16	1.98E-14	5.59E-13	7.86E-12
7	0	0	0	0	0	0	0	1.11E-16	0	0
8	0	0	0	0	0	0	0	5.55E-17	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0

Table 25: Approximated error of PECE with Exponential type example and Robin Boundary condition.

	NPECE		HPECE	
N	error	roc	error	roc
10	0.00674	-	0.00674	-
20	0.00269	1.32232	0.00269	1.32232
40	0.00104	1.37959	0.00104	1.37959
80	0.00039	1.40863	0.00039	1.40863
160	1.45E-04	1.42363	1.45E-04	1.42363
320	5.39E-05	1.43109	5.39E-05	1.43109

Table 26: Convergence rate of PECE methods with Exponential type example and Robin Boundary condition

	NHOM					HHOM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.41411	0.89055	1.42639	2.02066	2.67403	0.41405	0.89041	1.42614	2.02026	2.67342
2	0.02775	0.09509	0.19065	0.311	0.45621	0.03142	0.10795	0.21952	0.36568	0.54942
3	0.00018	0.00193	0.00707	0.01696	0.03265	0.00026	0.0028	0.01039	0.02545	0.05023
4	1.51E-08	9.56E-07	1.20E-05	6.71E-05	0.00024	1.61E-08	2.10E-06	2.89E-05	0.00017	0.00065
5	6.59E-13	4.20E-11	5.57E-10	3.99E-09	2.45E-08	1.25E-13	1.52E-11	1.33E-12	6.56E-09	1.08E-07
6	0	1.89E-15	2.44E-14	1.75E-13	1.07E-12	5.55E-17	1.67E-16	1.11E-16	5.11E-14	8.40E-13
7	0	0	0	0	1.67E-16	5.55E-17	5.55E-17	5.55E-17	5.55E-17	5.55E-17
8	0	0	0	0	5.55E-17	5.55E-17	5.55E-17	5.55E-17	5.55E-17	0
9	0	0	0	0	0	5.55E-17	5.55E-17	5.55E-17	5.55E-17	0
10	0	0	0	0	0	5.55E-17	5.55E-17	5.55E-17	5.55E-17	0

Table 27: Approximated error of HOMs with Exponential type example and Robin Boundary condition.

	NHOM		HHOM	
N	error	roc	error	roc
10	0.0011	-	5.65E-05	-
20	4.05E-04	1.43433	1.71E-05	1.72533
40	1.19E-04	1.76458	2.82E-06	2.59886
80	3.21E-05	1.89256	3.99E-07	2.82439
160	8.32E-06	1.94851	5.29E-08	2.91251
320	2.12E-06	1.9743	6.84E-09	2.95208

Table 28: Convergence rate of HOMs with Exponential type example and Robin Boundary condition.

NHM						HHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.41411	0.89055	1.42639	2.02066	2.67403	0.41405	0.89043	1.42617	2.02031	2.6735
2	0.02742	0.09412	0.18898	0.30872	0.45353	0.03143	0.108	0.21966	0.36596	0.5499
3	1.69E-04	1.86E-03	0.00686	0.01654	0.03195	0.00026	0.00281	0.01041	0.02549	0.05031
4	6.66E-09	8.04E-07	1.09E-05	6.23E-05	2.29E-04	1.81E-08	2.13E-06	2.90E-05	0.00017	0.00065
5	1.67E-16	1.72E-13	2.76E-11	9.03E-10	1.22E-08	5.55E-17	1.23E-12	2.29E-10	7.96E-09	1.14E-07
6	0	5.55E-17	0	5.55E-17	3.33E-16	5.55E-17	5.55E-17	5.55E-17	5.55E-17	3.50E-15
7	0	0	0	0	0	5.55E-17	5.55E-17	0	0	5.55E-17
8	0	0	0	0	0	5.55E-17	5.55E-17	0	0	5.55E-17
9	0	0	0	0	0	5.55E-17	5.55E-17	0	0	5.55E-17
10	0	0	0	0	0	5.55E-17	5.55E-17	0	0	5.55E-17

Table 29: Approximated error of HMs with Exponential type example and Robin Boundary condition.

N	NHM		HHM	
	error	roc	error	roc
10	0.00135	-	8.43E-05	-
20	4.38E-04	1.62581	1.88E-05	2.16215
40	1.23E-04	1.83097	2.92E-06	2.68841
80	3.25E-05	1.9189	4.04E-07	2.85479
160	8.37E-06	1.95937	5.32E-08	2.92421
320	2.12E-06	1.97881	6.85E-09	2.95677

Table 30: Convergence rate of HMs with Exponential type example and Robin Boundary condition.

NIHM						HIHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.41412	0.89055	1.4264	2.02067	2.67405	0.41405	0.89043	1.42617	2.02031	2.67349
2	0.02742	0.09412	0.18898	0.30872	0.45353	0.03143	0.108	0.21966	0.36596	0.54989
3	0.00017	0.00186	0.00686	0.01654	0.03196	0.00026	0.00281	0.01041	0.02549	0.05031
4	6.66E-09	8.04E-07	1.09E-05	6.23E-05	0.00023	1.81E-08	2.13E-06	2.90E-05	0.00017	0.00065
5	5.55E-17	1.39E-15	2.72E-11	9.01E-10	1.22E-08	1.67E-16	1.23E-12	2.29E-10	7.96E-09	1.14E-07
6	5.55E-17	5.55E-17	1.11E-16	5.55E-17	1.11E-16	1.11E-16	1.11E-16	1.11E-16	1.11E-16	3.55E-15
7	5.55E-17	5.55E-17	0	5.55E-17	5.55E-17	1.11E-16	1.11E-16	1.67E-16	1.11E-16	1.11E-16
8	5.55E-17	5.55E-17	0	5.55E-17	5.55E-17	1.11E-16	1.11E-16	1.11E-16	1.11E-16	1.67E-16
9	5.55E-17	5.55E-17	0	5.55E-17	5.55E-17	1.11E-16	1.11E-16	1.11E-16	1.11E-16	1.11E-16
10	5.55E-17	5.55E-17	0	5.55E-17	5.55E-17	1.11E-16	1.11E-16	1.11E-16	1.11E-16	1.11E-16

Table 31: Approximated error of IHMs with Exponential type example and Robin Boundary condition.

	NIHM		HIHM	
N	error	roc	error	roc
10	2.07E-03	-	0.00018	-
20	5.30E-04	1.96918	2.49E-05	2.85921
40	1.34E-04	1.97928	3.29E-06	2.92012
80	3.39E-05	1.9861	4.26E-07	2.94907
160	8.53E-06	1.99039	5.46E-08	2.96587
320	2.14E-06	1.99343	6.94E-09	2.97582

Table 32: Convergence rate of IHMs with Exponential type example and Robin Boundary condition.

Example 2.3.3. *Sin type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) - y(t)^2 + D_0^{\alpha_1} y(t), \\ a_1 y(0) + b_1 y'(0) = \gamma_1, \quad a_2 y(1) + b_2 y'(1) = \gamma_2. \end{cases} \quad (\text{II.50})$$

whose $\alpha_2 = 1.7$, $\alpha_1 = 0.4$,

$$\begin{aligned} F(t) = & -\frac{1}{2}i(i)^{[\alpha_2]}t^{([\alpha_2]-\alpha_2)}(E_{[\alpha_2]-\alpha_2+1}(it) - (-1)^{[\alpha_2]}E_{1,[\alpha_2]-\alpha_2+1}(-it)) \\ & + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)}t^{3-\alpha_2} + \left(\sin(t) - t + \frac{t^3}{6}\right)^2 \\ & + \frac{1}{2}i(i)^{[\alpha_1]}t^{([\alpha_1]-\alpha_1)}(E_{[\alpha_1]-\alpha_1+1}(it) - (-1)^{[\alpha_1]}E_{1,[\alpha_1]-\alpha_1+1}(-it)) \\ & + \frac{\Gamma(2)}{\Gamma(2-\alpha_1)}t^{1-\alpha_1} - \frac{\Gamma(4)}{6\Gamma(4-\alpha_1)}t^{3-\alpha_1}. \end{aligned}$$

and exact solution is

$$y(t) = \sin(t) - t + \frac{t^3}{6}. \quad (\text{II.51})$$

First, $a_1 = 1$, $b_1 = 0$, $a_2 = 1$, $b_2 = 0$. That case is called ‘Dirichlet Boundary Condition’.

	NPECE					HPECE				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.2823	0.54788	0.79772	1.03281	1.25406	0.2823	0.54788	0.79772	1.03281	1.25406
2	0.02554	0.09517	0.19987	0.33159	0.48335	0.01042	0.04038	0.08616	0.14272	0.20371
3	2.34E-04	2.97E-03	0.01288	0.03515	0.07425	2.53E-05	0.00026	0.00106	0.00278	0.00554
4	2.67E-07	5.98E-06	6.64E-05	4.30E-04	1.83E-03	2.71E-08	2.82E-07	1.27E-06	3.99E-06	9.94E-06
5	2.85E-10	6.39E-09	7.22E-08	5.17E-07	3.01E-06	2.89E-11	3.01E-10	1.36E-09	4.26E-09	1.06E-08
6	3.04E-13	6.82E-12	7.70E-11	5.52E-10	3.22E-09	3.09E-14	3.21E-13	1.45E-12	4.54E-12	1.13E-11
7	3.23E-16	7.28E-15	8.22E-14	5.89E-13	3.43E-12	3.12E-17	3.43E-16	1.55E-15	4.85E-15	1.21E-14
8	1.73E-18	6.94E-18	8.67E-17	6.28E-16	3.66E-15	0	1.73E-18	1.73E-18	5.20E-18	1.21E-17
9	0	0	1.73E-18	1.73E-18	3.47E-18	0	0	0	1.73E-18	1.73E-18
10	0	0	1.73E-18	1.73E-18	1.73E-18	0	0	0	1.73E-18	1.73E-18

Table 33: Approximated error of PECE with Sin type example and Dirichlet Boundary condition.

	NPECE		HPECE	
N	error	roc	error	roc
10	0.00054	-	0.00054	-
20	0.00022	1.3193	0.00022	1.3193
40	8.31E-05	1.38846	8.31E-05	1.38846
80	3.11E-05	1.42029	3.11E-05	1.42029
160	1.15E-05	1.43462	1.15E-05	1.43462
320	4.23E-06	1.4411	4.23E-06	1.4411

Table 34: Convergence rate of PECE methods with Sin type example and Dirichlet Boundary condition

	NHOM					HHOM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.2826	0.54824	0.79801	1.03294	1.25395	2.83E-01	5.48E-01	7.98E-01	1.03E+00	1.25E+00
2	0.02564	0.09552	0.20056	0.33264	0.48474	1.80E-02	6.96E-02	1.51E-01	2.58E-01	3.87E-01
3	0.00024	0.003	0.01299	0.03545	0.07484	8.90E-05	1.12E-03	5.14E-03	1.51E-02	3.42E-02
4	2.67E-07	6.03E-06	6.73E-05	0.00044	0.00186	9.67E-08	1.46E-06	1.11E-05	6.46E-05	2.87E-04
5	2.82E-10	6.39E-09	7.26E-08	5.22E-07	3.06E-06	1.03E-10	1.56E-09	1.19E-08	6.99E-08	3.24E-07
6	2.98E-13	6.75E-12	7.67E-11	5.52E-10	3.24E-09	1.10E-13	1.67E-12	1.27E-11	7.47E-11	3.46E-10
7	3.16E-16	7.13E-15	8.11E-14	5.83E-13	3.42E-12	1.16E-16	1.78E-15	1.36E-14	7.98E-14	3.69E-13
8	0	6.94E-18	8.50E-17	6.18E-16	3.62E-15	1.73E-18	1.73E-18	1.56E-17	8.50E-17	3.97E-16
9	0	0	0	0	3.47E-18	1.73E-18	0	1.73E-18	1.73E-18	1.73E-18
10	0	0	0	0	0	1.73E-18	0	1.73E-18	1.73E-18	1.73E-18

Table 35: Approximated error of HOMs with Sin type example and Dirichlet Boundary condition.

	NHOM		HHOM	
N	error	roc	error	roc
10	5.42E-05	-	1.14E-05	-
20	2.28E-05	1.25086	1.58E-06	2.84513
40	6.84E-06	1.73458	1.90E-07	3.05613
80	1.85E-06	1.88864	2.26E-08	3.07058
160	4.79E-07	1.94779	2.74E-09	3.04644
320	1.22E-07	1.97352	3.36E-10	3.02582

Table 36: Convergence rate of HOMs with Sin type example and Dirichlet Boundary condition.

	NHM					HHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.2826	0.54824	0.79801	1.03294	1.25395	0.28264	0.54832	0.79813	1.03309	1.25413
2	0.02581	0.09608	0.20163	0.33428	0.48692	0.01807	0.0697	0.15129	0.25868	0.38745
3	0.00024	0.00306	0.01321	0.03602	0.07596	8.93E-05	0.00112	0.00516	0.01511	0.03426
4	2.76E-07	6.26E-06	7.00E-05	0.00045	0.00193	9.73E-08	1.47E-06	1.12E-05	6.51E-05	0.00029
5	2.96E-10	6.72E-09	7.66E-08	5.52E-07	3.25E-06	1.04E-10	1.58E-09	1.20E-08	7.06E-08	3.27E-07
6	3.17E-13	7.20E-12	8.20E-11	5.91E-10	3.49E-09	1.12E-13	1.69E-12	1.29E-11	7.56E-11	3.50E-10
7	3.42E-16	7.71E-15	8.79E-14	6.33E-13	3.74E-12	1.20E-16	1.81E-15	1.38E-14	8.10E-14	3.75E-13
8	0	8.67E-18	9.37E-17	6.80E-16	4.00E-15	1.73E-18	1.73E-18	1.56E-17	8.67E-17	3.99E-16
9	0	0	1.73E-18	1.73E-18	3.47E-18	1.73E-18	1.73E-18	1.73E-18	1.73E-18	1.73E-18
10	0	0	1.73E-18	0	0	1.73E-18	1.73E-18	1.73E-18	1.73E-18	1.73E-18

Table 37: Approximated error of HMs with Sin type example and Dirichlet Boundary condition.

	NHM		HHM	
N	error	roc	error	roc
10	7.03E-05	-	1.17E-05	-
20	2.47E-05	1.5099	1.55E-06	2.92347
40	7.04E-06	1.81089	1.86E-07	3.0556
80	1.87E-06	1.91461	2.24E-08	3.05846
160	4.81E-07	1.95744	2.72E-09	3.03785
320	1.22E-07	1.97732	3.35E-10	3.02124

Table 38: Convergence rate of HMs with Sin type example and Dirichlet Boundary condition.

	NIHM					HIHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.2826	0.54824	0.79801	1.03294	1.25395	0.28264	0.54831	0.79813	1.03309	1.25413
2	0.02581	0.09608	0.20164	0.33429	0.48694	0.01807	0.0697	0.15129	0.25868	0.38745
3	0.00024	0.00306	0.01321	0.03602	0.07597	8.93E-05	0.00112	0.00516	0.01511	0.03425
4	2.77E-07	6.26E-06	7.00E-05	0.00045	0.00193	9.73E-08	1.47E-06	1.12E-05	6.51E-05	0.00029
5	2.96E-10	6.72E-09	7.66E-08	5.52E-07	3.25E-06	1.04E-10	1.58E-09	1.20E-08	7.06E-08	3.27E-07
6	3.17E-13	7.20E-12	8.21E-11	5.91E-10	3.49E-09	1.12E-13	1.69E-12	1.29E-11	7.56E-11	3.50E-10
7	3.40E-16	7.72E-15	8.79E-14	6.33E-13	3.74E-12	1.21E-16	1.81E-15	1.38E-14	8.10E-14	3.75E-13
8	0	1.04E-17	9.37E-17	6.80E-16	4.01E-15	1.73E-18	1.73E-18	1.56E-17	8.67E-17	4.02E-16
9	0	3.47E-18	1.73E-18	1.73E-18	5.20E-18	0	0	1.73E-18	1.73E-18	0
10	0	0	0	0	1.73E-18	0	0	1.73E-18	0	0

Table 39: Approximated error of IHMs with Sin type example and Dirichlet Boundary condition.

	NIHM		HIHM	
N	error	roc	error	roc
10	1.17E-04	-	1.13E-05	-
20	3.01E-05	1.96038	1.37E-06	3.03823
40	7.67E-06	1.97379	1.70E-07	3.01491
80	1.94E-06	1.98228	2.11E-08	3.00464
160	4.90E-07	1.98723	2.64E-09	3.00171
320	1.23E-07	1.99128	3.30E-10	3.00015

Table 40: Convergence rate of IHMs with Sin type example and Dirichlet Boundary condition.

Second, $a_1 = 1$, $b_1 = 1$, $a_2 = 1$, $b_2 = 1$. That case is called ‘Robin Boundary Condition’.

	NPECE					HPECE				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.48704	0.89047	1.20133	1.40555	1.4823	0.48704	0.89047	1.20133	1.40555	1.4823
2	0.10143	0.34375	0.65281	0.97315	1.25738	0.11394	0.37116	0.68561	1.00193	1.27576
3	0.00447	0.05055	0.18268	0.41346	0.72588	0.00659	0.06702	0.22204	0.4699	0.78405
4	1.26E-05	0.00113	0.01435	0.07309	0.22639	2.81E-05	0.00232	0.02448	0.10625	0.28875
5	1.19E-08	1.60E-06	0.0001	0.00234	0.02199	2.67E-08	4.89E-06	0.00033	0.00574	0.04099
6	1.11E-11	1.50E-09	9.87E-08	4.51E-06	0.00023	2.49E-11	4.58E-09	3.58E-07	2.20E-05	0.00089
7	1.04E-14	1.40E-12	9.23E-11	4.22E-09	2.33E-07	2.33E-14	4.28E-12	3.35E-10	2.08E-08	1.22E-06
8	6.94E-18	1.31E-15	8.63E-14	3.95E-12	2.18E-10	0	3.98E-15	3.13E-13	1.95E-11	1.15E-09
9	0	0	8.33E-17	3.71E-15	2.04E-13	0	0	3.05E-16	1.82E-14	1.07E-12
10	0	0	0	0	1.94E-16	0	0	1.39E-17	1.39E-17	1.01E-15

Table 41: Approximated error of PECE with Sin type example and Robin Boundary condition.

	NPECE		HPECE	
N	error	roc	error	roc
10	0.00119	-	0.00119	-
20	0.00048	1.31165	0.00048	1.31165
40	1.85E-04	1.3724	1.85E-04	1.3724
80	6.99E-05	1.40339	6.99E-05	1.40339
160	2.61E-05	1.41954	2.61E-05	1.41954
320	9.72E-06	1.42774	9.72E-06	1.42774

Table 42: Convergence rate of PECE methods with Sin type example and Robin Boundary condition

NHOM						HHOM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.48762	0.89157	1.20299	1.40781	1.48511	0.48772	0.89183	1.20347	1.40858	1.48624
2	0.10168	0.34427	0.65343	0.97388	1.2585	0.10626	0.35557	0.66898	0.98986	1.27077
3	0.00449	0.05073	0.18298	0.41358	0.72559	0.00519	0.05676	0.19931	0.44048	0.75788
4	1.26E-05	0.00114	0.01441	0.07316	0.2261	1.69E-05	0.00151	0.01801	0.08681	0.25589
5	1.13E-08	1.57E-06	0.0001	0.00234	0.02195	1.54E-08	2.39E-06	0.00016	0.00348	0.02956
6	1.01E-11	1.41E-09	9.49E-08	4.43E-06	0.00022	1.39E-11	2.16E-09	1.59E-07	8.62E-06	0.00042
7	9.06E-15	1.26E-12	8.48E-11	3.97E-09	2.22E-07	1.25E-14	1.95E-12	1.44E-10	7.82E-09	4.62E-07
8	6.94E-18	1.12E-15	7.58E-14	3.55E-12	1.99E-10	6.94E-18	1.76E-15	1.30E-13	7.06E-12	4.17E-10
9	0	1.39E-17	6.94E-17	3.16E-15	1.78E-13	6.94E-18	6.94E-18	1.25E-16	6.37E-15	3.77E-13
10	0	6.94E-18	6.94E-18	6.94E-18	1.67E-16	6.94E-18	0	6.94E-18	6.94E-18	3.33E-16

Table 43: Approximated error of HOMs with Sin type example and Robin Boundary condition.

NHOM			HHOM		
N	error	roc	error	roc	
10	1.73E-04	-	1.19E-05	-	
20	6.85E-05	1.33816	3.97E-06	1.57784	
40	2.07E-05	1.72711	6.62E-07	2.58621	
80	5.63E-06	1.87683	9.28E-08	2.83353	
160	1.47E-06	1.94177	1.22E-08	2.92648	
320	3.74E-07	1.97138	1.56E-09	2.96587	

Table 44: Convergence rate of HOMs with Sin type example and Robin Boundary condition.

NHOM						HHOM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.48762	0.89157	1.20299	1.4078	1.48509	0.48772	0.89182	1.20343	1.4085	1.48612
2	0.10144	0.34381	0.653	0.97366	1.25848	0.10628	0.35563	0.66908	0.98997	1.27085
3	0.00445	0.05045	0.18238	0.4129	0.72517	0.00519	0.05679	0.19941	0.44067	0.75816
4	1.24E-05	1.12E-03	1.43E-02	0.07273	0.22543	1.69E-05	0.00151	0.01803	0.0869	0.25613
5	1.13E-08	1.55E-06	9.92E-05	2.31E-03	2.18E-02	1.54E-08	2.40E-06	0.00016	0.00349	0.02962
6	1.02E-11	1.40E-09	9.36E-08	4.34E-06	2.20E-04	1.40E-11	2.18E-09	1.60E-07	8.66E-06	0.00042
7	9.17E-15	1.26E-12	8.44E-11	3.92E-09	2.19E-07	1.27E-14	1.97E-12	1.45E-10	7.88E-09	4.65E-07
8	0	1.14E-15	7.61E-14	3.54E-12	1.98E-10	6.94E-18	1.78E-15	1.32E-13	7.15E-12	4.22E-10
9	0	0	6.94E-17	3.18E-15	1.78E-13	1.39E-17	6.94E-18	1.18E-16	6.47E-15	3.82E-13
10	0	0	0	0	1.60E-16	1.39E-17	1.39E-17	1.39E-17	1.39E-17	3.33E-16

Table 45: Approximated error of HMs with Sin type example and Robin Boundary condition.

	NHM		HHM	
N	error	roc	error	roc
10	2.33E-04	-	2.23E-05	-
20	7.64E-05	1.60895	4.64E-06	2.2632
40	2.16E-05	1.82124	7.00E-07	2.72951
80	5.74E-06	1.91417	9.49E-08	2.883
160	1.48E-06	1.95714	1.23E-08	2.9457
320	3.75E-07	1.97776	1.57E-09	2.97362

Table 46: Convergence rate of HMs with Sin type example and Robin Boundary condition.

	NIHM					HIHM				
m	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0	s=0.2	s=0.4	s=0.6	s=0.8	s=1.0
1	0.48762	0.89157	1.203	1.40782	1.48512	0.48772	0.89181	1.20343	1.40849	1.4861
2	0.10144	0.34381	0.653	0.97365	1.25848	0.10628	0.35563	0.66908	0.98997	1.27084
3	0.00445	0.05045	0.18238	0.41289	0.72515	0.00519	0.05679	0.19941	0.44068	0.75816
4	1.24E-05	0.00112	0.01427	0.07272	0.22542	1.69E-05	0.00151	0.01804	0.0869	0.25613
5	1.13E-08	1.55E-06	9.92E-05	0.00231	0.02175	1.54E-08	2.40E-06	0.00016	0.00349	0.02962
6	1.02E-11	1.40E-09	9.36E-08	4.34E-06	0.00022	1.40E-11	2.18E-09	1.60E-07	8.66E-06	0.00042
7	9.15E-15	1.26E-12	8.44E-11	3.92E-09	2.19E-07	1.27E-14	1.97E-12	1.45E-10	7.88E-09	4.65E-07
8	6.94E-18	1.14E-15	7.61E-14	3.53E-12	1.96E-10	2.78E-17	1.78E-15	1.32E-13	7.15E-12	4.22E-10
9	0	1.39E-17	4.86E-17	3.20E-15	1.77E-13	6.94E-18	6.94E-18	1.11E-16	6.47E-15	3.82E-13
10	0	1.39E-17	6.94E-18	1.39E-17	1.53E-16	0	6.94E-18	6.94E-18	6.94E-18	3.61E-16

Table 47: Approximated error of IHMs with Sin type example and Robin Boundary condition.

	NIHM		HIHM	
N	error	roc	error	roc
10	3.66E-04	-	4.47E-05	-
20	9.35E-05	1.96851	6.05E-06	2.8839
40	2.37E-05	1.97858	7.85E-07	2.94664
80	5.99E-06	1.98542	1.00E-07	2.97273
160	1.51E-06	1.98992	1.26E-08	2.98554
320	3.79E-07	1.99312	1.59E-09	2.99187

Table 48: Convergence rate of IHMs with Sin type example and Robin Boundary condition.

PECEs seems to have $p = \min \{2, 1 - \alpha_2, 1 - \alpha_1\}$ convergence rate. And convergence rates of HOMs, HMs and AHM are 2 and 3 independently of α_2 and α_1 . Their main methods to solve the system comes from the following papers [14], [1]. Therefore, it can be seen that the convergence rate will be derived as the above result.

Due to structural of main solving scheme, Covnvergence rates of HOMs, HMs and IHMs are better then PECE's. And Convergence rate for HOMs, HMs and IHMs are looks similar, but, the approximated error is different. Dirichlet boundary condition examples exactly shows that HMs and IHMs are little better than HOMs. However approximated error of Robin boundary condition examples looks like similar each other.

To begin with, transformed system of FIVP have a special iterate structure. So, this paper suggest the numerical method using that constitution which is called "Iterated Hybrid Method". But, Numerical result show that methods don't have impressive efficiency than "Hybrid Method". Especially, Dirichlet boundary condition cases show us better results about approximated error than that of Robin boundary condition, but those things are not noticeable. And in numerical results of IHMs have slightly stable convergence rate, nevertheless, its error almost same to HMs. Maybe in iterate process, numerical error is accumulated at each time.

III Fractional Two-Point Boundary Value Problem with 1st order Derivative Term

And also below case can be considered.

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y(t), y'(t)) \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b}, \end{cases} \quad (\text{III.1})$$

where $1 < \alpha_2 < 2$, $\alpha_2 \in \mathbb{R}$ and g be linear function. Now, let's talk about how to solve above case. Below Figure.3 briefly show outline of solving process. But, there are two questions about

$$\begin{aligned} & \begin{cases} D_0^{\alpha_2} y(t) = f(t, y, y') \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b} \end{cases} \\ & \quad \downarrow ? \\ & \begin{cases} D_0^{\alpha_2} y(t) = f(t, y, D_0^{1-\epsilon} y), \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b} \end{cases} \\ & \quad \downarrow \\ & \begin{cases} D_0^{1-\epsilon} y(t) = w(t), & y(t_0) = y_a \\ D_0^\epsilon w(t) = z(t), & w(t_0) = w_a \\ D_0^{\alpha_2-1} z(t) = f(t, y, w), & z(t_0) = s \end{cases} \\ & \quad \downarrow ? \\ & \begin{cases} D_0^{1-\epsilon} y(t) = z(t), & y(t_0) = y_a \\ D_0^{\alpha_2-1} z(t) = f(t, y, z), & z(t_0) = s \end{cases} \end{aligned}$$

Figure 3: Structure of solver for BVP with ODE

above process. First, how to replace 1st derivative term to fractional one? The intuitive answer of the first question is that “almost 1” derivative nearly same as “1” derivative. Second, how to reduce the number of system? The intuitive answer is “almost 0” derivative virtually same “0” derivative that means “no derivative”. Thus, below contents will talk about that precisely.

3.1 Description of Fractional Boundary Value Problem containing 1st order Derivative

Let's answer first question.

Lemma 3.1.1. (First Gronwall inequality for two-term equations [10]).

Let $\alpha_2 > 0$ and $\alpha_1, \tilde{\alpha}_1 \in (0, \alpha_2)$ be chosen so that the equation

$$D_0^{\alpha_2} y(t) = f(t, y(t), D_0^{\alpha_1} y(t)), \quad (\text{III.2})$$

subject to be initial conditions

$$y(0) = y_0, \quad y'(0) = y'_0, \quad \dots, \quad y^{(\lceil \alpha_2 \rceil - 1)}(0) = y_0^{(\lceil \alpha_2 \rceil - 1)}$$

and

$$D_0^{\alpha_2} z(t) = f(t, z(t), D_0^{\tilde{\alpha}_1} \tilde{z}(t))$$

subject to the same initial condition

$$z(0) = y_0, \quad z'(0) = y'_0, \quad \dots, \quad z^{(\lceil \alpha_2 \rceil - 1)}(0) = y_0^{(\lceil \alpha_2 \rceil - 1)}$$

(where f satisfies a Lipschitz condition in its second and third arguments on a suitable domain) have unique continuous solutions $y, z : [0, T] \rightarrow \mathbb{R}$. We assume further that $[\alpha_1] = [\tilde{\alpha}_1]$. Then there exist constants K_1 and K_2 such that

$$|y(t) - z(t)| \leq K_1 |\alpha_1 - \tilde{\alpha}_1| E_{\alpha_n}(K_2 T^{\alpha_2}), \quad \forall t \in [0, T]. \quad (\text{III.3})$$

Thus, above equation (III.1) can be approximated to below equation by Lemma (3.1.1).

$$\begin{cases} D_a^{\alpha_2} y(t) = f(t, y(t), D_a^{1-\epsilon} y(t)) \\ g(y(t), y'(t)) = \gamma_t|_{t=a,b}, \end{cases} \quad (\text{III.4})$$

where $1 < \alpha_2 < 2$, $\epsilon \rightarrow 0+$ and g be linear function. α_2 can be variable order. Then, Two Point boundary value problem can be similarly transformed to system of initial value problem.

$$\begin{cases} D_a^{1-\epsilon} y(t) = w(t), & y(t_0) = y_a \\ D_a^\epsilon w(t) = z(t), & w(t_0) = 0 \\ D_a^{\alpha_2-1} z(t) = f(t, y(t), w(t)). & z(t_0) = s \end{cases} \quad (\text{III.5})$$

If ϵ is really small constant then w and z intuitively is almost same. Actually, there are examples (2.3.1), (2.3.2) and (2.3.3) whose data are shown the naive idea is in a measure correct. But, there isn't any guarantee then the idea needs proof.

Lemma 3.1.2. [10] Let $f \in C[a, b]$ and $m \geq 0$. Moreover assume that α_k is a sequence of positive numbers such that $\alpha_k \rightarrow \alpha$ as $k \rightarrow \infty$. Then, for every $\epsilon > 0$,

$$\lim_{k \rightarrow \infty} \sup_{t \in [a+\epsilon, b]} |J_a^{\alpha_k} f(t) - J_a^\alpha f(t)| = 0. \quad (\text{III.6})$$

Lemma 3.1.3. *Let, $0 < \gamma \leq \alpha \leq \beta$ the,*

$$|J_a^\alpha y(t) + J_a^\beta y(t)| \leq \left[\frac{\Gamma(\gamma)}{\Gamma(\alpha)} T^{\alpha-\gamma} + \frac{\Gamma(\gamma)}{\Gamma(\beta)} T^{\beta-\gamma} \right] J_a^\gamma |y(t)|. \quad (\text{III.7})$$

Proof.

$$\begin{aligned} J_a^\alpha y(t) + J_a^\beta y(t) &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} y(\tau) d\tau + \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\beta-1} y(\tau) d\tau \\ &= \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\gamma-1+(\alpha-\gamma)} y(\tau) d\tau + \frac{1}{\Gamma(\beta)} \int_a^t (t-\tau)^{\gamma-1+(\beta-\gamma)} y(\tau) d\tau \end{aligned} \quad (\text{III.8})$$

Then,

$$\begin{aligned} |J_a^\alpha y(t) + J_a^\beta y(t)| &\leq \frac{T^{\alpha-\gamma}}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\gamma-1} |y(\tau)| d\tau + \frac{T^{\beta-\gamma}}{\Gamma(\beta)} \int_a^t (t-\tau)^{\gamma-1} |y(\tau)| d\tau \\ &\leq \left[\frac{\Gamma(\gamma)}{\Gamma(\alpha)} T^{\alpha-\gamma} + \frac{\Gamma(\gamma)}{\Gamma(\beta)} T^{\beta-\gamma} \right] \frac{1}{\Gamma(\gamma)} \int_a^t (t-\tau)^{\gamma-1} |y(\tau)| d\tau \\ &\leq \left[\frac{\Gamma(\gamma)}{\Gamma(\alpha)} T^{\alpha-\gamma} + \frac{\Gamma(\gamma)}{\Gamma(\beta)} T^{\beta-\gamma} \right] J_a^\gamma |y(t)|. \end{aligned} \quad (\text{III.9})$$

□

Now, let's answer to Second question.

Lemma 3.1.4. *Let $1 < \alpha_2 < 2$, $\forall T \in \mathbb{R}^+$ and $f : [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ is satisfies a Lipschitz condition in its second and third arguments on a suitable domain. as*

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \leq L (|x_2 - x_1| + |y_2 - y_1|) \quad (\text{III.10})$$

where $\forall t \in [0, T]$, $x_1, x_2, y_1, y_2 : [0, T] \rightarrow \mathbb{R}$ and $0 < L$. if for any $0 < \epsilon \ll 1$ and \hat{y} and \tilde{y} are solutions of below systems,

$$\begin{cases} D_a^{1-\epsilon} \hat{y}(t) = w(t), & y(t_0) = y_a \\ D_a^\epsilon w(t) = z(t), & w(t_0) = 0 \\ D_a^{\alpha_2-1} z(t) = f(t, \hat{y}(t), w(t)). & z(t_0) = s \end{cases} \quad \text{and} \quad \begin{cases} D_a^{1-\epsilon} \tilde{y}(t) = \tilde{z}(t), & \tilde{y}(t_0) = y_a \\ D_a^{\alpha_2-1} \tilde{z}(t) = f(t, \tilde{y}(t), \tilde{z}(t)). & \tilde{z}(t_0) = s \end{cases} \quad (\text{III.11})$$

then,

$$|\hat{y}(t) - \tilde{y}(t)| \rightarrow 0, \quad \text{as } \epsilon \rightarrow 0. \quad (\text{III.12})$$

Proof.

$$\hat{y}(t) = y_a + J_a^{1-\epsilon} w(t), \quad \tilde{y}(t) = y_a + J_a^{1-\epsilon} \tilde{z}(t), \quad (\text{III.13})$$

$$\Rightarrow \hat{y}(t) - \tilde{y}(t) = J_a^{1-\epsilon} (w(t) - \tilde{z}(t)), \quad (\text{III.14})$$

$$w(t) - \tilde{z}(t) = w(t) - J_a^\epsilon \tilde{z}(t) + J_a^\epsilon \tilde{z}(t) - \tilde{z}(t). \quad (\text{III.15})$$

Since, $w(t) = J_a^\epsilon z(t)$.

$$\Rightarrow |w(t) - \tilde{z}(t)| \leq J_a^\epsilon |z(t) - \tilde{z}(t)| + |J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)|. \quad (\text{III.16})$$

Also, $z(t) = s + J_a^{\alpha_2-1}f(t, \hat{y}(t), w(t))$ and $\tilde{z}(t) = s + J_a^{\alpha_2-1}f(t, \tilde{y}(t), w(t))$.

$$\begin{aligned}
|z(t) - \tilde{z}(t)| &= |J_a^{\alpha_2-1}f(t, \hat{y}(t), w(t)) - f(t, \tilde{y}(t), \tilde{z}(t))| \\
&\leq J_a^{\alpha_2-1}|f(t, \hat{y}(t), w(t)) - f(t, \tilde{y}(t), \tilde{z}(t))| \\
&\leq LJ_a^{\alpha_2-1}(|\hat{y}(t) - \tilde{y}(t)| + |w(t) - \tilde{z}(t)|) \quad \text{by Lipschitz condition.} \\
&\leq LJ_a^{\alpha_2-1}[J_a^{1-\epsilon}|w(t) - \tilde{z}(t)| + J_a^\epsilon|z(t) - \tilde{z}(t)| + |J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)|] \\
&\quad \text{by (III.14) and (III.16).} \\
&\leq L\left[J_a^{\alpha_2-\epsilon}|w(t) - \tilde{z}(t)| + J_a^{\alpha_2-1+\epsilon}|z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)}\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty\right]. \tag{III.17}
\end{aligned}$$

Now, we need to calculate $J_a^{\alpha_2-\epsilon}|w(t) - \tilde{z}(t)|$.

$$\begin{aligned}
J_a^{\alpha_2-\epsilon}|w(t) - \tilde{z}(t)| &\leq J_a^{\alpha_2-\epsilon}(J_a^\epsilon|z(t) - \tilde{z}(t)| + |J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)|) \\
&\leq J_a^{\alpha_2}|z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)}\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty \tag{III.18}
\end{aligned}$$

Using (III.18), we have

$$\begin{aligned}
|z(t) - \tilde{z}(t)| &\leq L\left[J_a^{\alpha_2}|z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)}\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty\right. \\
&\quad \left.+ J_a^{\alpha_2-1+\epsilon}|z(t) - \tilde{z}(t)| + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)}\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty\right] \\
&= L\left[J_a^{\alpha_2-\epsilon}|z(t) - \tilde{z}(t)| + J_a^{\alpha_2-1+\epsilon}|z(t) - \tilde{z}(t)|\right. \\
&\quad \left.+ \left\{\frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)}\right\}\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty\right] \\
&\quad \text{by Lemma (3.1.3), let } \gamma = \alpha_2 - 1, \alpha = \alpha_2 - 1 + \epsilon, \text{ and } \beta = \alpha_2 \text{ then,} \\
&\leq L\left[\left\{\frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2 - 1 + \epsilon)}T^\epsilon + \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2)}T\right\}J_a^{\alpha_2-1}|z(t) - \tilde{z}(t)|\right. \\
&\quad \left.+ \left\{\frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)}\right\}\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty\right] \tag{III.19}
\end{aligned}$$

Let,

$$C_\epsilon^1 \equiv \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2 - 1 + \epsilon)}T^\epsilon + \frac{\Gamma(\alpha_2 - 1)}{\Gamma(\alpha_2)}T, \quad C_\epsilon^2 \equiv \frac{T^{\alpha_2-\epsilon}}{\Gamma(\alpha_2 - \epsilon + 1)} + \frac{T^{\alpha_2-1}}{\Gamma(\alpha_2)} \tag{III.20}$$

So, we have

$$|z(t) - \tilde{z}(t)| \leq LC_\epsilon^2\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty + LC_\epsilon^1J_a^{\alpha_2-1}|z(t) - \tilde{z}(t)|. \tag{III.21}$$

Applying the Lemma 6.19 of [10],

$$|z(t) - \tilde{z}(t)| \leq LC_\epsilon^2\|J_a^\epsilon\tilde{z}(t) - \tilde{z}(t)\|_\infty E_{\alpha_2-1}[LC_\epsilon^1T^{\alpha_2-1}]. \tag{III.22}$$

Therefore,

$$\begin{aligned}
|\tilde{y}(t) - \hat{y}(t)| &\leq J_a^{1-\epsilon} |w(t) - \tilde{z}(t)| \\
&\leq J_a |z(t) - \tilde{z}(t)| + J_a^{1-\epsilon} |J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)| \\
&\leq T |z(t) - \tilde{z}(t)| + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)} \|J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)\|_\infty \\
&\leq TLC_\epsilon^2 \|J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)\|_\infty E_{\alpha_2-1}[LC_\epsilon^1 T^{\alpha_2-1}] + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)} \|J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)\|_\infty \\
&= \left[TLC_\epsilon^2 E_{\alpha_2-1}[LC_\epsilon^1 T^{\alpha_2-1}] + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)} \right] \|J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)\|_\infty
\end{aligned} \tag{III.23}$$

Let, $C_\epsilon \equiv TLC_\epsilon^2 E_{\alpha_2-1}[LC_\epsilon^1 T^{\alpha_2-1}] + \frac{T^{1-\epsilon}}{\Gamma(2-\epsilon)}$. Then,

$$|\tilde{y}(t) - \hat{y}(t)| \leq C_\epsilon \|J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)\|_\infty. \tag{III.24}$$

Thus, by Lemma (3.1.2).

$$|\tilde{y}(t) - \hat{y}(t)| \leq C_\epsilon \|J_a^\epsilon \tilde{z}(t) - \tilde{z}(t)\|_\infty \rightarrow 0, \text{ as } \epsilon \rightarrow 0. \tag{III.25}$$

□

3.2 Numerical Results

Bellow tables and figures guarantee correctness of Lemma (3.1.4). Those show that the maximum error go to zero when epsilon decrease to zero also. In this section HM is used.

Example 3.2.1. *Polynomial type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = \frac{\Gamma(4-\alpha_2)}{\Gamma(5-\alpha_2)} t^{4-\alpha_2} - \frac{\Gamma(8)}{\Gamma(8-\alpha_2)} t^{7-\alpha_2} - \exp(t^4) + \exp(y) - \sin(4t^3) + \sin(y') \\ y(0) - \frac{1}{\alpha_2-1} y'(0) = \gamma_1, \quad y(1) + y'(1) = \gamma_2. \end{cases} \quad (\text{III.26})$$

whose $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \dots, 10$ and exact solution is

$$y(t) = t^4. \quad (\text{III.27})$$

i	NHM	HHM
1	0.050506	0.05124
2	0.004765	0.004734
3	0.000545	0.00047
4	0.000136	4.74E-05
5	9.63E-05	5.11E-06
6	9.25E-05	1.14E-06
7	9.21E-05	8.54E-07
8	9.20E-05	8.26E-07
9	9.20E-05	8.23E-07
10	9.20E-05	8.23E-07

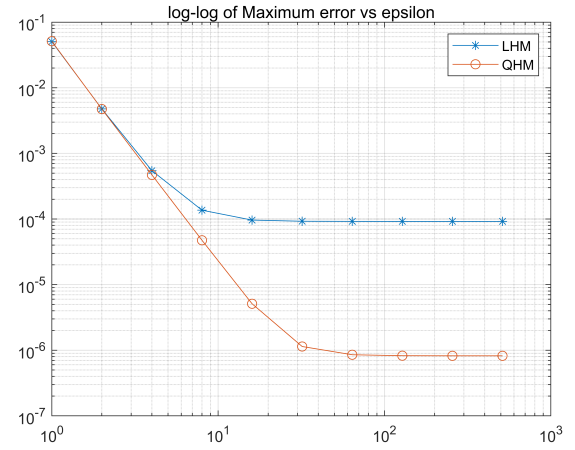


Figure 4: Maximum error vs ϵ in Example.(3.2.1)

Example 3.2.2. *exponential type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) + \exp(y) + \sin(y'), \\ y(0) - \frac{1}{\alpha_2 - 1} y'(0) = \gamma_1, \quad y(1) + y'(1) = \gamma_2. \end{cases} \quad (\text{III.28})$$

where $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \dots, 10$,

$$\begin{aligned} F(t) = & t^{2-\alpha_2} E_{1,3-\alpha_2}(t) - \left(\frac{\Gamma(3)}{2\Gamma(3-\alpha_2)} t^{2-\alpha_2} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)} t^{3-\alpha_2} \right) \\ & - \exp \left(e^t - \left(1 + t + \frac{1}{2} t^2 + \frac{1}{3!} t^3 \right) \right) - \sin \left(e^t - \left(1 + t + \frac{1}{2} t^2 \right) \right), \end{aligned}$$

and exact solution is

$$y(t) = e^t - \left(1 + t + \frac{1}{2} t^2 + \frac{1}{3!} t^3 \right). \quad (\text{III.29})$$

i	NHM	HHM
1	0.004241	0.004194
2	0.000403	0.000401
3	3.79E-05	3.99E-05
4	7.52E-06	3.97E-06
5	6.21E-06	3.87E-07
6	6.08E-06	6.91E-08
7	6.06E-06	5.99E-08
8	6.06E-06	5.90E-08
9	6.06E-06	5.89E-08
10	6.06E-06	5.89E-08

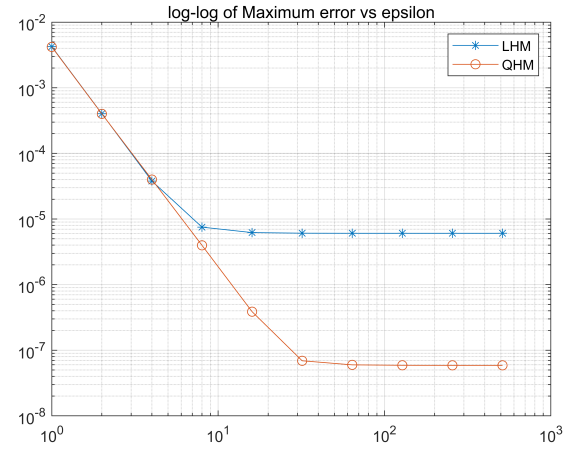


Figure 5: Maximum error vs ϵ in Example.(3.2.2)

Example 3.2.3. *Sin type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) + \exp(y) + \sin(y'), \\ y(0) - \frac{1}{\alpha_2 - 1} y'(0) = \gamma_1, \quad y(1) + y'(1) = \gamma_2. \end{cases} \quad (\text{III.30})$$

whose $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \dots, 10$,

$$\begin{aligned} F(t) = & -\frac{1}{2} i(i)^{[\alpha_2]} t^{([\alpha_2] - \alpha_2)} (E_{[\alpha_2] - \alpha_2 + 1}(it) - (-1)^{[\alpha_2]} E_{1, [\alpha_2] - \alpha_2 + 1}(-it)) \\ & - \exp\left(e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3\right)\right) - \sin\left(e^t - \left(1 + t + \frac{1}{2}t^2\right)\right), \end{aligned}$$

and exact solution is

$$y(t) = \sin(t) - t + \frac{t^3}{6}. \quad (\text{III.31})$$

i	NHM	HHM
1	0.000763	0.000757
2	7.16E-05	7.18E-05
3	6.43E-06	7.13E-06
4	1.83E-06	7.09E-07
5	1.60E-06	6.70E-08
6	1.58E-06	1.89E-08
7	1.58E-06	1.68E-08
8	1.58E-06	1.66E-08
9	1.58E-06	1.65E-08
10	1.58E-06	1.65E-08

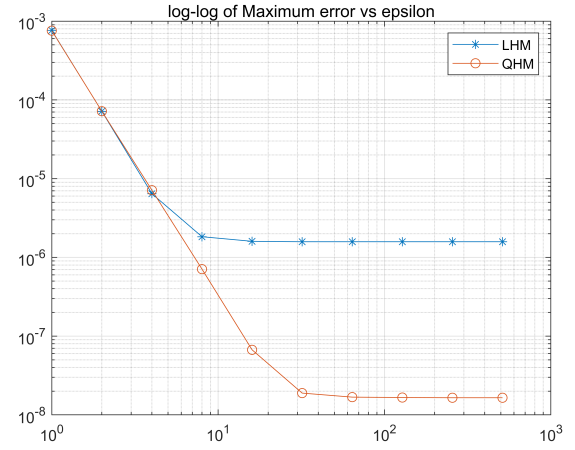


Figure 6: Maximum error vs ϵ in Example.(3.2.3)

Next numerical results will compare between conventional method [15] and new schemes(NHOM and HHOM with 1st Derivative).

Example 3.2.4. *Single-term Fractional Orders, Robin BCs, C^1 solution*

$$\begin{cases} D_0^\alpha y(t) = F(t) - (2t + 6)y'(t) \\ y(0) - \frac{1}{1-\alpha}y'(0) = \gamma_1, \quad y(1) + y'(1) = \gamma_2 \end{cases} \quad (\text{III.32})$$

whose $1 < \alpha < 2$, and

$$F(t) = \frac{\Gamma(\alpha+1)}{\Gamma(1)} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}t^{(\alpha-1)} + 4\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{(3-\alpha)} + \frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{(4-\alpha)} + (2t + 6)(\alpha x^{\alpha-1} + (2\alpha - 1)t^{2\alpha-2} + 3 + 12t^2 + 3t^2).$$

and exact solution is

$$y(t) = x^\alpha + t^{2\alpha-1} + 1 + 3t + 4t^3 + t^4. \quad (\text{III.33})$$

In numerical algorithm, $F(t)$ term is calculated as analysis and $\epsilon = 10^{-10}$. And in this example, initial vale s_0 means $y(0)$.

MID	$\alpha = 1.1$		$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
N	error	roc	error	roc	error	roc	error	roc	error	roc
64	9.56E-03	-	5.14E-03	-	3.91E-03	-	3.76E-03	-	4.07E-03	-
128	3.89E-03	1.298	1.60E-03	1.686	1.05E-03	1.895	9.44E-04	1.995	1.02E-03	1.996
256	1.65E-03	1.236	5.20E-04	1.618	2.87E-04	1.872	2.37E-04	1.997	2.55E-04	1.998
512	7.22E-04	1.193	1.78E-04	1.548	8.01E-05	1.840	5.92E-05	1.998	6.39E-05	1.999
1024	3.22E-04	1.165	6.36E-05	1.484	2.30E-05	1.803	1.51E-05	1.971	1.60E-05	1.999
2048	1.45E-04	1.148	2.36E-05	1.432	6.77E-06	1.762	3.90E-06	1.955	4.00E-06	2.000
NHOM	$\alpha = 1.1$		$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
N	error	roc	error	roc	error	roc	error	roc	error	roc
64	4.42E-03	-	1.31E-03	-	1.14E-03	-	1.27E-03	-	1.42E-03	-
128	1.89E-03	1.225	3.83E-04	1.775	2.95E-04	1.955	3.17E-04	1.999	0.00035	2.000
256	8.10E-04	1.224	1.27E-04	1.588	7.62E-05	1.955	7.92E-05	2.002	8.87E-05	2.000
512	3.48E-04	1.219	6.31E-05	1.012	1.97E-05	1.949	1.98E-05	2.004	2.22E-05	2.000
1024	1.50E-04	1.213	2.88E-05	1.130	5.15E-06	1.938	4.92E-06	2.006	5.54E-06	2.000
2048	6.91E-05	1.120	1.26E-05	1.191	1.36E-06	1.922	1.22E-06	2.008	1.38E-06	2.001

Table 49: Convergence rate of A MID and transformed NHOM in Example (3.2.4). Initial value : MID ($\beta_1 = \gamma_1, \beta_2 = \gamma_2$), NHOM ($s_0 = 0.2$)

Example 3.2.5. *Single-term Fractional Orders, Robin BCs, C^∞ solution*

$$\begin{cases} D_0^\alpha y(t) = F(t) - \cos(t)y(t) - \sin(t)y'(t) \\ y(0) - \frac{1}{1-\alpha}y'(0) = \gamma_1, \quad y(1) + y'(1) = \gamma_2 \end{cases} \quad (\text{III.34})$$

whose $1 < \alpha < 2$ and exact solution is

$$y(t) = \sin(\lambda t) - t + \frac{t^3}{6}. \quad (\text{III.35})$$

In numerical algorithm $\epsilon = 10^{-10}$. And in this example, initial vale s_0 means $y(0)$.

MID	$\alpha = 1.1$		$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
N	error	roc	error	roc	error	roc	error	roc	error	roc
10	3.36E-04	-	3.42E-04	-	3.81E-04	-	4.18E-04	-	4.48E-04	-
20	8.57E-05	1.971	9.08E-05	1.911	1.03E-04	1.880	1.14E-04	1.871	1.23E-04	1.869
40	2.13E-05	2.010	2.33E-05	1.966	2.69E-05	1.942	2.99E-05	1.936	3.21E-05	1.936
80	5.20E-06	2.033	5.84E-06	1.993	6.86E-06	1.973	7.63E-06	1.969	8.20E-06	1.968
160	1.26E-06	2.045	1.45E-06	2.005	1.73E-06	1.988	1.93E-06	1.984	2.07E-06	1.984
320	3.04E-07	2.051	3.61E-07	2.010	4.34E-07	1.995	4.85E-07	1.992	5.21E-07	1.992
NHOM	$\alpha = 1.1$		$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
N	error	roc	error	roc	error	roc	error	roc	error	roc
10	7.27E-03	-	1.97E-03	-	9.45E-04	-	6.80E-04	-	5.80E-04	-
20	1.78E-03	2.033	3.92E-04	2.329	1.97E-04	2.263	1.56E-04	2.124	1.40E-04	2.047
40	3.92E-04	2.180	7.80E-05	2.328	4.38E-05	2.170	3.75E-05	2.055	3.47E-05	2.016
80	8.38E-05	2.226	1.62E-05	2.272	1.03E-05	2.093	9.25E-06	2.021	8.64E-06	2.005
160	1.78E-05	2.236	3.50E-06	2.205	2.49E-06	2.046	2.30E-06	2.007	2.16E-06	2.001
320	3.79E-06	2.232	7.93E-07	2.144	6.12E-07	2.021	5.74E-07	2.002	5.39E-07	2.000
HHOM	$\alpha = 1.1$		$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
N	error	roc	error	roc	error	roc	error	roc	error	roc
10	1.08E-03	-	2.39E-04	-	9.60E-05	-	6.16E-05	-	5.08E-05	-
20	1.19E-04	3.185	2.20E-05	3.440	9.55E-06	3.329	7.10E-06	3.117	6.34E-06	3.001
40	1.26E-05	3.239	2.09E-06	3.398	1.04E-06	3.198	8.60E-07	3.044	7.98E-07	2.991
80	1.32E-06	3.254	2.08E-07	3.325	1.21E-07	3.105	1.06E-07	3.015	1.00E-07	2.992
160	1.39E-07	3.253	2.19E-08	3.248	1.46E-08	3.052	1.33E-08	3.004	1.26E-08	2.995
320	1.47E-08	3.245	2.42E-09	3.179	1.79E-09	3.024	1.66E-09	3.000	1.58E-09	2.997

Table 50: Convergence rate of MID, NHOM and HHOM in Example (3.2.5). Initial value : MID ($\beta_1 = \gamma_1, \beta_2 = \gamma_2$), NHOM ($s_0 = 0.2$), HHOM ($s_0 = 0.2$)

Example 3.2.6. *exponential type example*

$$\begin{cases} D_0^{\alpha_2} y(t) = F(t) + y^2 + ty', \\ y(0) - \frac{1}{\alpha_2 - 1} y'(0) = \gamma_1, \quad y(1) + y'(1) = \gamma_2. \end{cases} \quad (\text{III.36})$$

where $\alpha_2 = 1.6$, $\epsilon = 10^{-i}$, $i = 1, 2, \dots, 10$,

$$\begin{aligned} F(t) = & t^{2-\alpha_2} E_{1,3-\alpha_2}(t) - \left(\frac{\Gamma(3)}{2\Gamma(3-\alpha_2)} t^{2-\alpha_2} + \frac{\Gamma(4)}{6\Gamma(4-\alpha_2)} t^{3-\alpha_2} \right) \\ & - \left(e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right) \right)^2 - t \left(e^t - \left(1 + t + \frac{1}{2}t^2 \right) \right), \end{aligned}$$

and exact solution is

$$y(t) = e^t - \left(1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 \right). \quad (\text{III.37})$$

In numerical algorithm $\epsilon = 10^{-10}$. In this example, initial value s_0 means $y(0)$.

NHOM	$\alpha = 1.1$		$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
N	error	roc	error	roc	error	roc	error	roc	error	roc
10	1.15E-02	-	1.65E-03	-	5.81E-04	-	9.99E-04	-	1.03E-03	-
20	3.02E-03	1.921	2.27E-04	2.862	2.49E-04	1.222	2.87E-04	1.801	2.67E-04	1.952
40	7.12E-04	2.086	1.69E-05	3.747	7.80E-05	1.675	7.58E-05	1.919	6.49E-05	2.039
80	1.59E-04	2.161	1.23E-05	0.461	2.18E-05	1.840	1.94E-05	1.965	1.70E-05	1.932
160	3.54E-05	2.171	3.83E-06	1.682	5.78E-06	1.915	4.91E-06	1.984	4.26E-06	1.998
320	8.58E-06	2.044	9.48E-07	2.013	1.49E-06	1.957	1.23E-06	1.991	1.07E-06	1.999
HHOM	$\alpha = 1.1$		$\alpha = 1.3$		$\alpha = 1.5$		$\alpha = 1.7$		$\alpha = 1.9$	
N	error	roc	error	roc	error	roc	error	roc	error	roc
10	6.18E-04	-	1.60E-04	-	4.84E-05	-	8.78E-05	-	9.77E-05	-
20	7.41E-05	3.061	1.01E-05	3.986	9.74E-06	2.312	1.48E-05	2.572	1.44E-05	2.765
40	8.43E-06	3.134	9.69E-07	3.379	1.69E-06	2.523	2.08E-06	2.827	1.93E-06	2.897
80	9.36E-07	3.171	1.50E-07	2.688	2.45E-07	2.791	2.76E-07	2.916	2.51E-07	2.941
160	1.02E-07	3.192	2.11E-08	2.832	3.29E-08	2.897	3.56E-08	2.953	3.23E-08	2.960
320	1.11E-08	3.206	2.83E-09	2.901	4.27E-09	2.946	4.54E-09	2.970	4.13E-09	2.968

Table 51: Convergence rate of A NHOM and transformed HHOM in Example (3.2.6). Initial value : NHOM ($s_0 = 0.2$) and HHOM ($s_0 = 0.2$).

IV Application

To transform fixed order α_2 and α_1 to variable order $\alpha_2(t)$ and $\alpha_1(t)$ was tried. That means $\alpha_2(t)$ and $\alpha_1(t)$ are some function defined on $[a, b] \rightarrow \mathbb{R}$. And there are some proper result when $\alpha_2(t)$ is defined by $[a, b] \rightarrow (1, 2)$, and α_1 defined by constant function. Then, that specific variable order case similarly works just like fixed order case. That means,

$$\begin{cases} D_a^{\alpha_2(t)} y(t) = f(t, y(t), D_a^{\alpha_1(t)} y(t)) \\ g(y(t), y'(t)) = \gamma t|_{t=a,b}, \end{cases} \quad (\text{IV.1})$$

where $1 < \alpha_2(t) < 2$, $0 < \alpha_1(t) < 1$, $\alpha_1 \in \mathbb{R}$, $\alpha_2(t) : [a, b] \rightarrow \mathbb{R}$ and g be linear function.

By similar process, above variable order two-point boundary value problem can be transformed to a variable order system of initial value problem, and also it can use Newton's and Halley's shooting method similarly.

But, if $\alpha_1(t)$ is not constant function, then it doesn't work properly. It will show bellow tables and figures, but it doesn't have any theoretical base. Also the general $\alpha_2(t)$ and $\alpha_1(t)$ case can be considered.

$$\begin{cases} D_a^{\alpha_2(t)} y(t) = f(t, y(t), D_a^{\alpha_1(t)} y(t)) \\ g(y(t), y'(t)) = \gamma t|_{t=a,b}, \end{cases} \quad (\text{IV.2})$$

where $\alpha_2(t) : [a, b] \rightarrow \mathbb{R}$, $\alpha_1(t) : [a, b] \rightarrow \mathbb{R}$ and g be linear function.

But, that case doesn't consider in this paper, thus, probably that will be future works.

4.1 Variable order Fractional Two-Point Boundary Value Problem

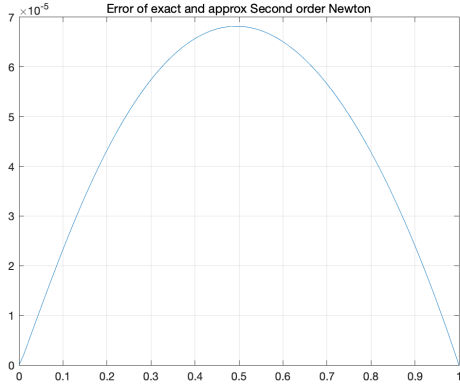
Bellow four examples will show good result of variable order BVP with Accelerated Hybrid Method. Shooting iteration m is 10 and $s = 0.2$.

Example 4.1.1. *Dirichlet Boundary Condition with variable order $\alpha_2(t)$.*

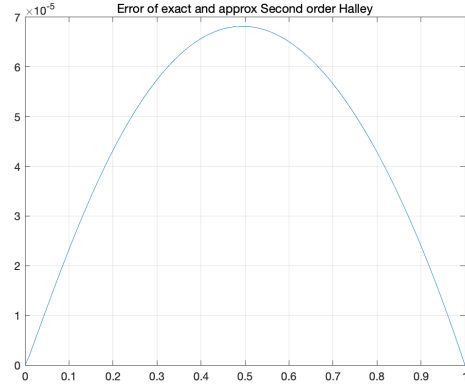
$$\begin{cases} D_0^{\alpha_2(t)} y(t) = \frac{\Gamma(5)}{\Gamma(5-\alpha_2(t))} t^{4-\alpha_2(t)} - t^8 + y^2 - \frac{\Gamma(5)}{\Gamma(5-\alpha_1(t))} t^{4-\alpha_1(t)} + D_0^{\alpha_1(t)} y(t) \\ y(0) = 0, \quad y(1) = 1, \end{cases} \quad (\text{IV.3})$$

whose $\alpha_2(t) = 1.0 + 0.99 \sin(\pi t/2)$, $\alpha_1(t) = 0.5$. and exact solution is

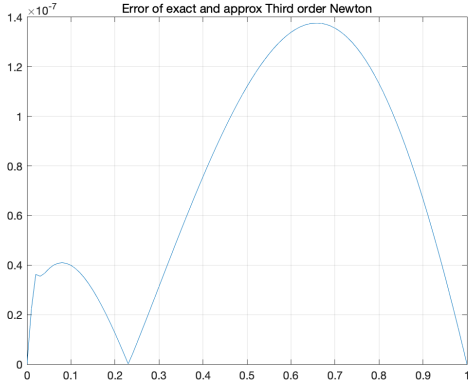
$$y(t) = y^4. \quad (\text{IV.4})$$



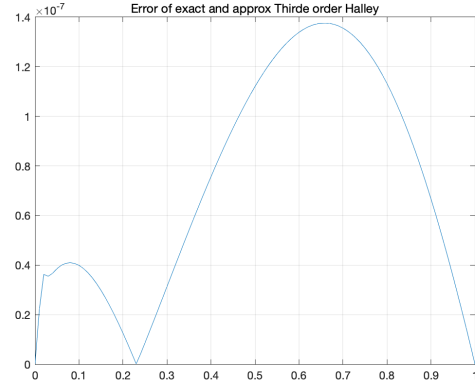
(a) SN Variable order with Dirichlet Boundary Condition



(b) SH Variable order with Dirichlet Boundary Condition



(c) SN Variable order with Dirichlet Boundary Condition



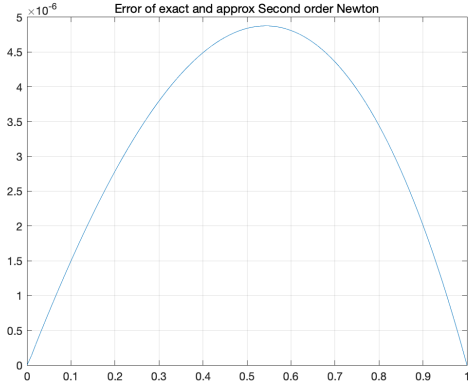
(d) SH Variable order with Dirichlet Boundary Condition

Example 4.1.2. *Robin Boundary Condition with variable order $\alpha_2(t)$.*

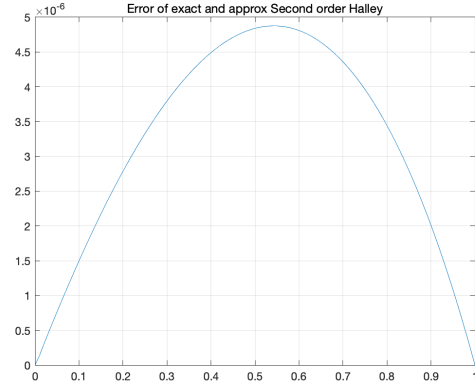
$$\begin{cases} D_0^{\alpha_2(t)} y(t) = \frac{\Gamma(5)}{\Gamma(5-\alpha_2(t))} t^{4-\alpha_2(t)} + \frac{\Gamma(5)}{\Gamma(5-\alpha_1)} t^{4-\alpha_1} + t^8 - y^2 - D_0^{\alpha_1} y(t) \\ y(0) + y'(0) = 0, \quad y(1) + y'(1) = 5 \end{cases} \quad (\text{IV.5})$$

whose $\alpha_2 = 1.0 + 0.99 \sin(\pi t/2)$, $\alpha_1 = 0.5$. and exact solution is

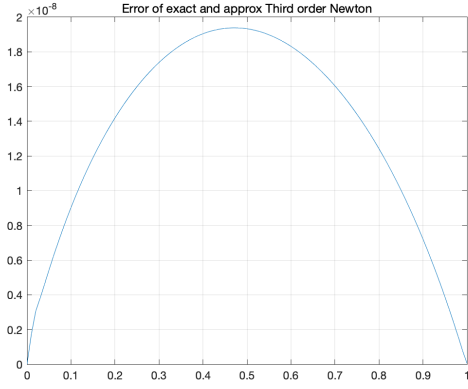
$$y(t) = y^4. \quad (\text{IV.6})$$



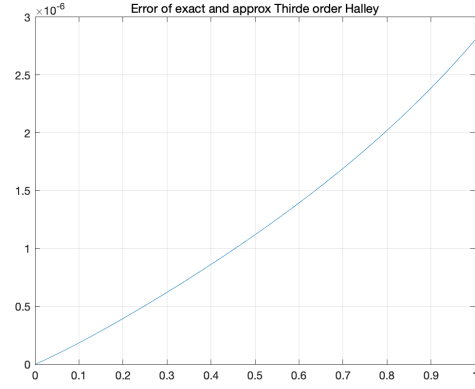
(a) SN Variable order with Robin Boundary Condition



(b) SH Variable order with Robin Boundary Condition



(c) SN Variable order with Robin Boundary Condition



(d) SH Variable order with Robin Boundary Condition

Example 4.1.3. *Dirichlet Boundary Condition with variable order $\alpha_2(t)$*

$$\begin{cases} D_0^{\alpha_2(t)} y(t) = \lambda^2 t^{2-\alpha_2(t)} E_{1,3-\alpha_2(t)}(\lambda t) - \left(\lambda^2 \frac{\Gamma(3)}{2\Gamma(3-\alpha_2(t))} t^{2-\alpha_2(t)} + \lambda^3 \frac{\Gamma(4)}{6\Gamma(4-\alpha_2(t))} t^{3-\alpha_2(t)} \right) \\ \quad - A^2 + y^2 - tB + tD_0^{\alpha_1} y(t), \\ y(t_0) = 0, \quad y(T) \approx 1.0577 \end{cases} \quad (\text{IV.7})$$

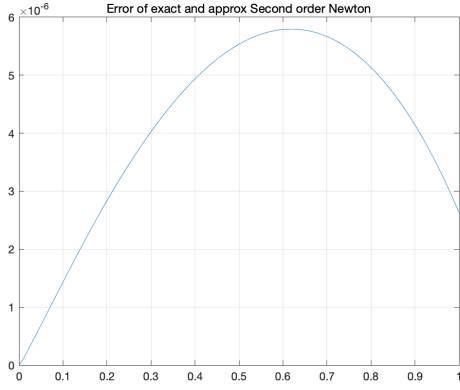
whose $\alpha_2(t) = 1.0 + 0.99 \sin(\pi t/2)$, $\alpha_1 = 0.5$, $\lambda = 2$,

$$A = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2} t^2 + \frac{\lambda^3}{3!} t^3 \right),$$

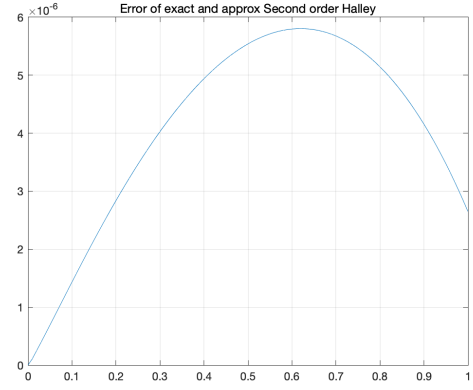
$$B = \lambda t^{1-\alpha_1} E_{1,2-\alpha_1}(\lambda t) - \left(\lambda \frac{\Gamma(2)}{\Gamma(2-\alpha_1)} t^{1-\alpha_1} + \lambda^2 \frac{\Gamma(3)}{2\Gamma(3-\alpha_1)} t^{2-\alpha_1} + \lambda^3 \frac{\Gamma(4)}{6\Gamma(4-\alpha_1)} t^{3-\alpha_1} \right)$$

and exact solution is

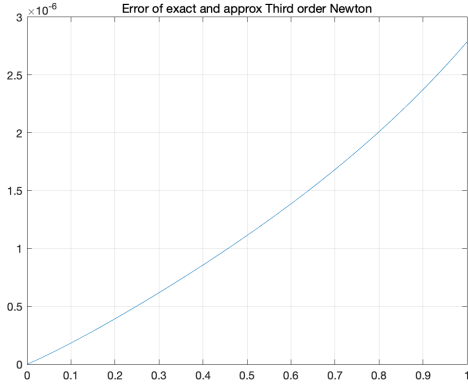
$$y(t) = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2} t^2 + \frac{\lambda^3}{3!} t^3 \right). \quad (\text{IV.8})$$



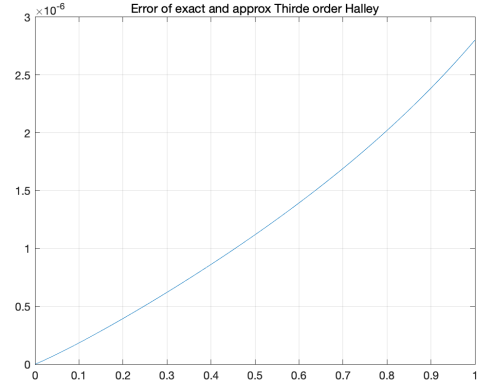
(a) SN Variable order with Dirichlet Boundary Condition



(b) SH Variable order with Dirichlet Boundary Condition



(c) SN Variable order with Dirichlet Boundary Condition



(d) SH Variable order with Dirichlet Boundary Condition

Example 4.1.4. *Robin Boundary Condition with variable order $\alpha_2(t)$*

$$\begin{cases} D_0^{\alpha_2(t)} y(t) = \lambda^2 t^{2-\alpha_2(t)} E_{1,3-\alpha_2(t)}(\lambda t) - \left(\lambda^2 \frac{\Gamma(3)}{2\Gamma(3-\alpha_2(t))} t^{2-\alpha_2(t)} + \lambda^3 \frac{\Gamma(4)}{6\Gamma(4-\alpha_2(t))} t^{3-\alpha_2(t)} \right) \\ \quad - A^2 + y^2 - tB + tD_0^{\alpha_1} y(t), \\ y(0) + y'(0) = 0, \quad y(1) + y'(1) = 5 \end{cases} \quad (\text{IV.9})$$

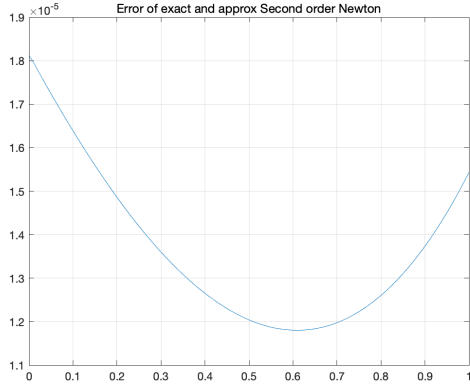
whose $\alpha_2(t) = 1.0 + 0.99 \sin(\pi t/2)$, $\alpha_1 = 0.5$, $\lambda = 2$,

$$A = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2} t^2 + \frac{\lambda^3}{3!} t^3 \right),$$

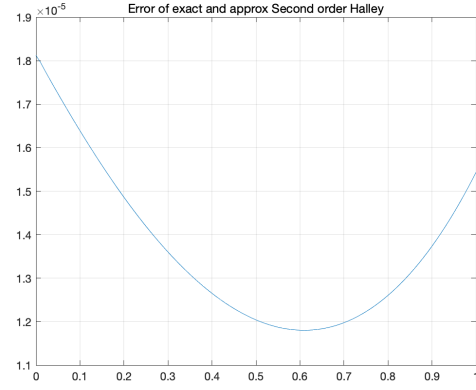
$$B = \lambda t^{1-\alpha_1} E_{1,2-\alpha_1}(\lambda t) - \left(\lambda \frac{\Gamma(2)}{\Gamma(2-\alpha_1)} t^{1-\alpha_1} + \lambda^2 \frac{\Gamma(3)}{2\Gamma(3-\alpha_1)} t^{2-\alpha_1} + \lambda^3 \frac{\Gamma(4)}{6\Gamma(4-\alpha_1)} t^{3-\alpha_1} \right)$$

and exact solution is

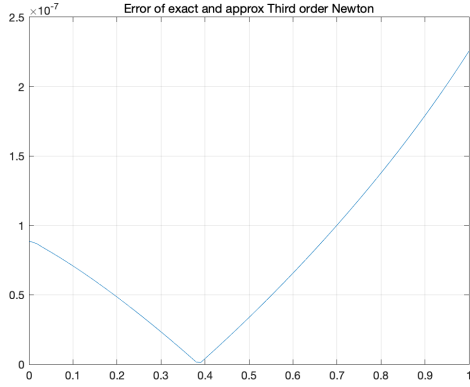
$$y(t) = e^{\lambda t} - \left(1 + \lambda t + \frac{\lambda^2}{2} t^2 + \frac{\lambda^3}{3!} t^3 \right). \quad (\text{IV.10})$$



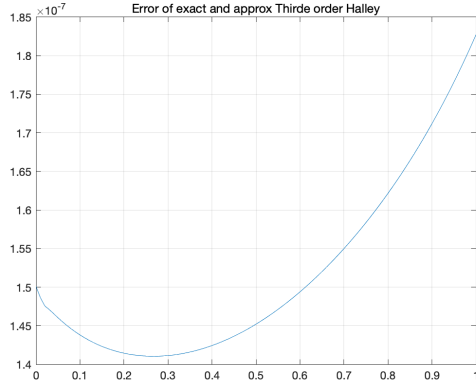
(a) SN Variable order with Robin Boundary Condition



(b) SH Variable order with Robin Boundary Condition



(c) SN Variable order with Robin Boundary Condition



(d) SH Variable order with Robin Boundary Condition

V Conclusion

In this paper, we suggest several more numerical approaches for solving FBVP under Caputo operator.

Process : The main ideas are described as follows

1. The given FBVP can be transposed into the system of FIVP.
2. Nonlinear shooting methods are applied to find unknown initial value for system of FIVP.
3. Four numerical methods are applied to approximated the system of FIVP.

Convergence rate of three methods which are HOM, HM and IHM is not depended under the differentiation order. But that of PECE is affected by the fractional order and is worse than the former three methods. And some of numerical results from HMs and IHMs are better than those obtained by HOMs, but they are almost the same and that is not remarkable.

And we also deal with FBVP with 1st order derivative term case.

Process :

1. Exchange 1st order term of FBVP to fractional order $\alpha_1 = 1 - \epsilon$ term which ϵ is very small

number.

2. Converted FBVP can be solved above process.
3. System of FIVP which is transposed from FBVP have ϵ fractional order term and it can be omitted.

Numerical results of FBVP with 1st order derivative term show that maximum error between exact value and approximated solution gets smaller as α_1 gets closer to 1. And suggested method can also solve examples which is come from conventional paper [15] and if the smoothness of given function is guaranteed, then the new method have better convergence rate than MID. Furthermore, conventional scheme can not solve nonlinear problem but, suggested methods can solve the examples.

Finally, although any theoretical consideration is absent and only intuitive idea from fixed order theoretical contents is present, numerical examples of variable order case are included. And the results show that generous interpretation works. Thus, that case must need precise theories, because that numerical results can be just a coincidence.

References

- [1] T. B. Nguyen and B. Jang, “A high-order predictor-corrector method for solving nonlinear differential equations of fractional order,” *Fractional Calculus and Applied Analysis*, vol. 20, no. 2, pp. 447–476, 2017.
- [2] Z. U. A. Zafar, M. Mushtaq, and K. Rehan, “A non-integer order dengue internal transmission model,” *Advances in Difference Equations*, vol. 2018, no. 1, p. 23, 2018.
- [3] E. Balci, İ. Öztürk, and S. Kartal, “Dynamical behaviour of fractional order tumor model with caputo and conformable fractional derivative,” *Chaos, Solitons & Fractals*, vol. 123, pp. 43–51, 2019.
- [4] S. Qureshi and A. Yusuf, “Modeling chickenpox disease with fractional derivatives: From caputo to atangana-baleanu,” *Chaos, Solitons & Fractals*, vol. 122, pp. 111–118, 2019.
- [5] F. Pelap, G. Tanekou, C. Fogang, and R. Kengne, “Fractional-order stability analysis of earthquake dynamics,” *Journal of Geophysics and Engineering*, vol. 15, no. 4, pp. 1673–1687, 2018.
- [6] S. Aman, I. Khan, Z. Ismail, M. Z. Salleh, and I. Tlili, “A new caputo time fractional model for heat transfer enhancement of water based graphene nanofluid: An application to solar energy,” *Results in physics*, vol. 9, pp. 1352–1362, 2018.
- [7] B. Yu, X. Jiang, and C. Wang, “Numerical algorithms to estimate relaxation parameters and caputo fractional derivative for a fractional thermal wave model in spherical composite medium,” *Applied Mathematics and Computation*, vol. 274, pp. 106–118, 2016.
- [8] S.-Y. Ha and J. Jung, “Remarks on the slow relaxation for the fractional kuramoto model for synchronization,” *Journal of Mathematical Physics*, vol. 59, no. 3, p. 032702, 2018.
- [9] M. Pakdaman, A. Ahmadian, S. Effati, S. Salahshour, and D. Baleanu, “Solving differential equations of fractional order using an optimization technique based on training artificial neural network,” *Applied Mathematics and Computation*, vol. 293, pp. 81–95, 2017.
- [10] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. Springer Science & Business Media, 2010.

- [11] M. Caputo and M. Fabrizio, “A new definition of fractional derivative without singular kernel,” *Progr. Fract. Differ. Appl.*, vol. 1, no. 2, pp. 1–13, 2015.
- [12] A. Atangana and D. Baleanu, “New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model,” *arXiv preprint arXiv:1602.03408*, 2016.
- [13] A. N. Fall, S. N. Ndiaye, and N. Sene, “Black–scholes option pricing equations described by the caputo generalized fractional derivative,” *Chaos, Solitons & Fractals*, vol. 125, pp. 108–118, 2019.
- [14] K. Diethelm, N. J. Ford, and A. D. Freed, “A predictor-corrector approach for the numerical solution of fractional differential equations,” *Nonlinear Dynamics*, vol. 29, no. 1-4, pp. 3–22, 2002.
- [15] Z. Cen, J. Huang, A. Xu, and A. Le, “A modified integral discretization scheme for a two-point boundary value problem with a caputo fractional derivative,” *Journal of Computational and Applied Mathematics*, vol. 367, p. 112465, 2020.

Acknowledgements

